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Asymptotic Calculus	
Western PA ARML Practice	*April 2, 2017*

1 Asymptotic notation

1.1 Definitions

Given two functions $f(x)$ and $g(x)$, we say that $f(x) \ll g(x)$ as $x \to \infty$ if any inequality of the form

$$|f(x)| \leq 0.000\ldots001 \cdot |g(x)|$$

eventually holds if you take x large enough. (How large you need x to be may depend on the number of zeroes in $0.000\ldots001$.) We can also write this as $g(x) \gg f(x)$ as $x \to \infty$.

We write $o(f(x))$ in an expression as shorthand for "some function $g(x)$, whose exact form isn't important, such that $f(x) \gg g(x)$". For example, we have $\binom{n}{3} = \frac{n^3}{6} + o(n^3)$ as $n \to \infty$, hiding lower-order terms with n^2 and n.

We write $f(x) \sim g(x)$ as $x \to \infty$ if $f(x) = g(x) + o(g(x))$. For example, $\binom{n}{3} \sim \frac{n^3}{6}$ as $n \to \infty$. The statement $f(x) \sim g(x)$ can also be written as $f(x) = g(x)(1 + o(1))$, and is equivalent to saying that any inequality of the form

$$0.999\ldots999 \leq \frac{f(x)}{g(x)} 1.000\ldots001$$

holds if you take x is large enough. (Again, how large you need x to be may depend on the number of zeroes or nines you want in this inequality.)

We can also say "$f(x) \ll g(x)$) as $x \to a$". This are defined in the same way, except that a relationship of the form

$$|f(x)| \leq 0.000\ldots001 \cdot |g(x)|$$

must hold not if x is *large enough*, but if x is *close enough to a*. We define "$f(x) = o(g(x))$ as $x \to a$ or "$f(x) \sim g(x)$ as $x \to a$" in terms of $\ll$, in the same way as we did for $x \to \infty$.

1.2 Practice

1. For what constant C is $\binom{n}{5} \sim Cn^5$ true as $n \to \infty$?
2. Show that there is some N such that whenever $n > N$,

   (a) n^2 exceeds $1000n$.

   (b) 2^n exceeds n^2.

   (c) 2^n exceeds n^3.

   (d) 2^n exceeds n^{100}.

   (e) 2^n exceeds $1000 \cdot n^{100}$.

   (f) $2^{n^{0.01}}$ exceeds n.

3. Verify as many as you like of the following statements, for $x \to \infty$:

$$1 \ll \log x \ll (\log x)^{100} \ll x^{0.01} \ll \sqrt{x} \ll x \ll x^2 \ll 2^x \ll 3^x \ll x! \ll x^x.$$

4. The n^{th} harmonic number H_n is equal to the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
   (a) Given $H_n = \log n + \gamma + o(1)$ as $n \to \infty$, write down an expression for $(H_n)^2$ with an $o(\log n)$ error term.
   (b) Given $H_n = \log n + \gamma + \frac{1}{2n} + o(\frac{1}{n})$ as $n \to \infty$, write down an expression for $(H_n)^2$ with the best error term you can.
   (c) Write down an expression for $H_{2n} - H_n$ with the best error term you can.

5. Prove that if $a_n \sim b_n$ as $n \to \infty$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} b_n \text{ converges.}$$

2 Derivatives

2.1 Definition

For a fixed value x (and function f), if there is a constant C such that, as $h \to 0$,

$$f(x+h) = f(x) + Ch + o(h),$$

then we say that C is *the derivative of f at x*, denoted $C = f'(x)$. Equivalent ways to say this using other notation:

$$f(x+h) - f(x) \sim f'(x) \cdot h \qquad f(x+h) - (f(x) + f'(x) \cdot h) \ll h$$

In other words, near a point a, we can approximate $f(x)$ by $f(a) + f'(a)(x-a)$.

2.2 Practice

1. Prove the product rule: if $h(x) = f(x) \cdot g(x)$, then $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.
2. Prove the chain rule: if $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$.
3. Prove that, if $f'(x) \neq 0$, then there is a value y (near x) such that $f(y) > f(x)$.

   (The converse of this statement is that if you want to find the largest value of $f(x)$, you figure out when $f'(x) = 0$.)
4. Prove that if $f(x) = x^n$ for an integer $n > 0$, then $f'(x) = nx^{n-1}$.
5. Using the fact that $e^x \sim x + 1$ as $x \to 0$, prove that if $f(x) = e^x$, then $f'(x) = e^x$.
6. Prove that $f(x) = |x|$ does not have a derivative at 0.

3 More Problems

1. Without finding a formula for $1^3 + 2^3 + \cdots + n^3$, prove that there is a constant C such that

$$1^3 + 2^3 + \cdots + n^3 \sim Cn^4$$

as $n \to \infty$. What can you say about C?

2. Find a function $f(x)$ such that as $x \to \infty$, *both* of the following hold:

   - $f(x) \gg x^n$ for any integer n;
   - $f(x) \ll a^x$ for any real number $a > 1$.

3. (a) Use the inequality $\sin x \leq x \leq \tan x$ (ask C.J. if you want a proof) to show that $\sin x \sim x$ as $x \to 0$.

   (b) Use the statement above to prove that if $f(x) = \sin x$ then $f'(x) = \cos x$.

4. (a) Prove that as $n \to \infty$, $\log n! \sim n \log n$.

   (b) Prove that if $n = k!$, then as $n \to \infty$, $k \sim \frac{\log n}{\log \log n}$.

   (c) A more precise estimate of $n!$ is given by Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty.$$

   Use this to prove that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$ as $n \to \infty$.

5. The Chebyshev function $\vartheta(x)$ is defined as

$$\vartheta(x) = \sum_{p \leq x} \log p$$

where the sum ranges over only *prime* p between 2 and x. The prime-counting function $\pi(x)$ is just the number of primes less than or equal to x.

Assuming that $\vartheta(x) \sim x$ as $x \to \infty$, prove that $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

6. Prove that the sum $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ of the reciprocals of the primes diverges.

7. Let P_n be the set of all n-digit palindromes. For example:

$$\begin{aligned} P_1 &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ P_2 &= \{11, 22, 33, 44, 55, 66, 77, 88, 99\} \\ P_3 &= \{101, 111, 121, \ldots, 979, 989, 999\} \end{aligned}$$

Find the best asymptotic estimate that you can of the sum

$$S_n = \sum_{p \in P_n} \frac{1}{p}$$

as $n \to \infty$.