Probability | Misha Lavrov

Random Variables

Western PA ARML Practice | *April 17, 2016*

1 Definitions

Given a probability space $(\Omega, \Pr)$, a *random variable* X is a function $X : \Omega \to \mathbb{R}$.

The *expected value* of X is

$$\mathbb{E}[X] := \sum_{\omega \in \Omega} X(\omega) \Pr(\omega).$$

If $\text{range}(X)$ is the set of all values X can take on, then we also have

$$\mathbb{E}[X] = \sum_{k \in \text{range}(X)} k \cdot \Pr[X = k].$$

2 Theoretical Exercises

1. Prove the following fact: if X is a random variable that takes on nonnegative integer values, then
$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \Pr[X \geq k].$$

2. A biased coin lands heads with probability p, and tails otherwise. We flip the coin until it lands heads for the first time; the random variable X is the number of times we flipped the coin.
   (a) Write an expression for $\Pr[X = k]$ in terms of p and k.
   (b) Compute $\mathbb{E}[X]$. (Intuitively, what should it be? Now prove it.)

3. Now take the same biased coin that lands heads with probability p, and toss it n times. Let Y be the number of times the coin lands heads.
   (a) Write an expression for $\Pr[Y = k]$ in terms of n, p, and k.
   (b) Compute $\mathbb{E}[Y]$. (Intuitively, what should it be? Now prove it.)

4. Let $X_1, X_2, \ldots, X_n$ be the outcomes of n independent rolls of fair six-sided dice. Let $S_k = X_1 + X_2 + \cdots + X_k$. Prove that $S_n \bmod 6$ has the same distribution as a single die roll.[1]

[1] Hint: Begin by proving this claim conditioned on $S_{n-1} = s$ for a fixed s.

5. (Found on Reddit) Three identical dice are contained in a transparent cube so that by throwing the cube, you can roll all three dice simultaneously.

   If you just want the outcome of a *single* die roll, then by the previous problem, you can take the total of the dice mod 6.

   Now find a method to convert a single throw of the cube into a value between 2 and 12 that has the same distribution as the sum of two dice. (Note that "take the sum of the first two dice, ignoring the third" is invalid: the dice are identical, so you have no way to distinguish one as the third die.)

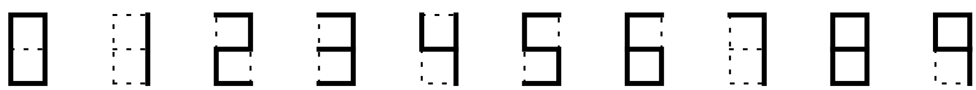
3 Other Problems

1. (ARML 1987) Initial setup: we have 3 jars, called A, B, and C. Jars A and B each contain one white and one black ball; jar C is empty. A random ball is chosen from jar A and placed into jar C. Similarly, a random ball is chosen from jar B and placed into jar C.

   We now consider jar C. A ball is chosen from jar C at random; it is white. That ball is put back into jar C, which is shaken, and again a ball is chosen at random; again it is white. That ball is put back into jar C, which is shaken, and for a third time a ball is chosen at random; again it is white.

   Compute the probability that the ball still in jar C is black.

2. (ARML 1999) A digital watch displays the digits from 0 to 9 as shown below. If one of the seven segments, randomly chosen, fails to light up, compute the expected value of the number of digits which can still be displayed.

3. The entire surface of a $3\times3\times3$ cube is painted, and then the cube is cut into 27 $1\times1\times1$ cubes. The small cubes are reassembled randomly into a $3\times3\times3$ cube. Compute the expected value of the number of 1×1 squares of paint showing anywhere on the resulting cube's surface.

4. (Inspired by AIME 2010) Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After going through security, which is at one end of the airport, Dave walks to his gate, and finds out that there's been a gate change, so Dave walks to his new gate (also assigned at random).

   Compute the expected value of the distance Dave has to walk.

5. Suppose that 100 fair coins are flipped simultaneously.

   (a) What is the expected number of *pairs of coins* that both land heads? (If 5 coins land heads and 95 coins land tails, then there are $\binom{5}{2} = 10$ pairs of coins that land heads.)

   (b) Let X be the number of coins that land heads. What is $\mathbb{E}[X^2]$?