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Coordinate Geometry

Western PA ARML Practice — *November 8, 2015*

Warm-up

1. *(ARML 2007) In rectangle $ABCD$, M is the midpoint of AB, AC and DM intersect at E, $CE = 10$, and $EM = 4$. Find the area of rectangle $ABCD$.*

   Let $A = (0, q)$, $B = (p, q)$, $C = (p, 0)$, and $D = (0, 0)$. Then $M = (\frac{p}{2}, q)$, and line DM has equation $y = \frac{2q}{p}x$, while line AC has equation $y = -\frac{q}{p}x + q$. Solving these two equations simultaneously, we get $x = \frac{p}{3}$ and $y = \frac{2q}{3}$, so point E has coordinates $(\frac{p}{3}, \frac{2q}{3})$.

   Then the distance formula says $16 = EM^2 = (\frac{p}{6})^2 + (\frac{q}{3})^2$, while $100 = EC^2 = (\frac{2p}{3})^2 + (\frac{2q}{3})^2$. Solving *these* two equations simultaneously gives $p = 6\sqrt{3}$ and $q = 3\sqrt{13}$, so the area of rectangle $ABCD$ is $pq = 18\sqrt{39}$.

Problems

1. *(ARML 1993) Triangle AOB is positioned in the first quadrant with $O = (0, 0)$ and B above and to the right of A. The slope of OA is 1, the slope of OB is 8, and the slope of AB is m. If the points A and B have x-coordinates a and b, respectively, compute $\frac{b}{a}$ in terms of m.*

   We have $A = (a, a)$ and $B = (b, 8b)$, respectively, from knowing the slopes of OA and OB. Then the slope of AB is
   $$m = \frac{8b - a}{b - a} = \frac{8\frac{b}{a} - 1}{\frac{b}{a} - 1} = \frac{8x - 1}{x - 1},$$
   where $x = \frac{b}{a}$. Solving for x, we get $m(x - 1) = 8x - 1$, or $(m - 8)x = m - 1$, or $x = \frac{m-1}{m-8}$.

2. *(ARML 1993) Square $ABCD$ is positioned in the first quadrant with A on the y-axis, B on the x-axis, and $C = (13, 8)$. Compute the area of the square.*

   Let $A = (0, y)$ and $B = (x, 0)$. Then the slope of AB is $-\frac{y}{x}$, so the slope of BC is $\frac{x}{y}$. Additionally, the lengths of AB and BC are equal; going from B the length of AB in the direction with slope $\frac{x}{y}$ gets us to $C = (x + y, x)$.

   But we know $C = (13, 8)$, s $x = 8$ and $x + y = 13$, giving us $y = 5$. The area of square $ABCD$ is $AB^2 = x^2 + y^2 = 8^2 + 5^2 = 89$.

3. (a) *Find the center of the circle that passes through the points $(3, 0)$, $(5, 12)$, and $(11, 11)$.*

   One of the easiest approaches uses the fact that the perpendicular bisector of a chord passes through the center of the circle.

The line through $(3,0)$ and $(5,12)$ has slope $\frac{12}{2} = 6$, so its perpendicular bisector has slope $-\frac{1}{6}$ and passes through its midpoint $(4,6)$: such a line has equation $y - 6 = -\frac{1}{6}(x-4)$.

The line through $(5,12)$ and $(11,11)$ has slope $-\frac{1}{6}$, so its perpendicular bisector has slope 6 and passes through its midpoint $(\frac{17}{2}, \frac{23}{2})$; such a line has equation $y - \frac{23}{2} = 6(x - \frac{17}{2})$.

We could simultaneously solve these two equations, but we also notice that the two slopes we found are negative reciprocals. So the lines are perpendicular, and the three points form a right triangle, meaning that the line through $(3,0)$ and $(11,11)$ is a diameter of the circle. Therefore the center of the circle is at their midpoint, $(7, \frac{11}{2})$.

(b) *Find the equation of the line tangent to this circle at* $(5,12)$.

The slope of the radius from $(7, \frac{11}{2})$ to $(5,12)$ is $-\frac{13}{4}$; the tangent line will be perpendicular to the radius, so it has slope $\frac{4}{13}$. So the tangent line has equation $y - 12 = \frac{4}{13}(x-5)$.

(c) *Another circle with center at* $(7,5)$ *is tangent to the first circle. Find the equation of the second circle, in the form* $(x-a)^2 + (y-b)^2 = c$.

We know the center of the second circle; the distance from $(7, \frac{11}{2})$ to $(7,5)$ is $\frac{1}{2}$, and the radius of the first circle was $\frac{\sqrt{185}}{2}$, so the radius of the second circle is $\frac{\sqrt{185}-1}{2}$, giving the equation

$$(x-7)^2 + (y-5)^2 = \left(\frac{\sqrt{185}-1}{2}\right)^2.$$

4. *(AIME 2000) Let u and v be integers satisfying $0 < v < u$. Let $A = (u,v)$, let B be the reflection of A across the line $y = x$, let C be the reflection of B across the y-axis, let D be the reflection of C across the x-axis, and let E be the reflection of D across the y-axis. The area of pentagon $ABCDE$ is 451. Find $u+v$.*

   We have $B = (v,u)$, $C = (-v,u)$, $D = (-v,-u)$, and $E = (v,-u)$. By the shoelace formula, the area of $ABCDE$ is $\frac{1}{2}(u^2 + 3uv + v^2) - \frac{1}{2}(v^2 - 3uv - u^2) = u^2 + 3uv$.

   So $u^2 + 3uv = u(u+3v) = 451$. Since both u and $u+3v$ are integers and $u + 3v > u$, then u and $u+3v$ must be factors of 451: either 1 and 451, or 11 and 41. But if $u = 1$ and $u + 3v = 451$, then $v = 150$, violating the condition that $v < u$. So we must have $u = 11$ and $u + 3v = 41$, giving $v = 10$ and $u + v = 21$.

5. *(AIME 2001) Let $R = (8,6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of PQ. The length of PQ equals $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.*

   We have $P = (8p, 15p)$ and $Q = (10q, 3q)$ for some unknown p, q. Taking the midpoint of thse, we get $R = (4p + 5q, \frac{15}{2}p + \frac{3}{2}q)$, but we also know $R = (8,6)$, giving us two equations for p and q. Solving them, we get $p = \frac{4}{7}$ and $q = \frac{8}{7}$, so $P = (\frac{32}{7}, \frac{60}{7})$ and $Q = (\frac{80}{7}, \frac{24}{7})$. The distance formula tells us $PQ = \frac{60}{7}$.

6. *Diameters AB and CD of circle S are perpendicular; E is another point on circle S. Chord EA intersects diameter CD at point K and chord EC intersects diameter AB at point L. If $CK : KD = 2 : 1$, find $AL : LB$.*

Let $A = (-1, 0)$, $B = (1, 0)$, $C = (0, -1)$, $D = (0, 1)$. Then $E = (p, q)$ such that $p^2 + q^2 = 1$; since EA intersects the y-axis (CD) and EC intersects the x-axis (AB), we know E is in the first quadrant.

If $CK : KD = 2 : 1$, then $K = (0, \frac{1}{3})$; EA passes through $(-1, 0)$ and $(0, \frac{1}{3})$, so it has slope $\frac{1}{3}$ and equation $y = \frac{1}{3}x + \frac{1}{3}$. Solving this together with $x^2 + y^2 = 1$ gives us $E = (\frac{4}{5}, \frac{3}{5})$. (The second solution corresponds to point A.)

Then the line EC passes through $(0, -1)$ and $(\frac{4}{5}, \frac{3}{5})$, so it has slope 2 and equation $y = 2x - 1$. Point L is on this line and has $y = 0$, so it has $x = \frac{1}{2}$. This divides the line from $(-1, 0)$ to $(1, 0)$ in a $3 : 1$ ratio.

7. *(ARML 1988 Power Round)*

   (a) *A sequence (x_n) is defined as follows: $x_0 = 2$, and for all $n \geq 1$, $(x_n, 0)$ lies on the line through $(0, 4)$ and $(x_{n-1}, 2)$. Derive a formula for x_n in terms of x_{n-1}.*

   As x goes from 0 to x_{n-1}, the line from $(0, 4)$ to $(x_{n-1}, 2)$ goes down by 2; it will go down by 2 again as x goes from x_{n-1} to $2x_{n-1}$, so $x_n = 2x_{n-1}$.

   (b) *A sequence (y_n) is defined as follows: $y_0 = 0$, and for all $n \geq 1$, draw a square of side length 2 with its bottom left corner at $(y_{n-1}, 0)$ and its bottom side on the x-axis. The point $(y_n, 0)$ lies on the line through $(0, 4)$ and the top right corner of the square. Derive a formula for y_n in terms of y_{n-1}.*

   The line in question must pass through $(0, 4)$ and $(y_{n-1} + 2, 2)$, so by the same logic as in part (a), $y_n = 2(y_{n-1} + 2)$.

   (c) *A sequence (z_n) is defined as follows: $z_0 = 0$, and for all $n \geq 1$, draw a circle of diameter 2 tangent to the x-axis and tangent to the line through $(0, 4)$ and $(z_{n-1}, 0)$ in such a way that its center lies to the right of that line. The line through $(0, 4)$ and $(z_n, 0)$ is the other tangent to the same circle. Derive a formula for z_n in terms of z_{n-1}.*

   The area of the triangle with vertices $(0, 4)$, $(z_{n-1}, 0)$, and $(z_n, 0)$ can be found in two ways. First, it is $2(z_n - z_{n-1})$ by the usual half-base-times-height formula. Second, it is the semi-perimeter times the radius of the inscribed circle (which is 1):

   $$1 \cdot \frac{1}{2}(z_n - z_{n-1} + \sqrt{16 + z_{n-1}^2} + \sqrt{16 + z_n^2}).$$

   Some tedious arithmetic after setting these equal yields $z_n = \frac{5z_{n-1} + 3\sqrt{16 + z_{n-1}^2}}{4}$.

   (d) *Express (x_n), (y_n), and (z_n) explicitly as functions of n.*

   We have $x_n = 2^{n+1}$ fairly easily.

   Next, $y_n + 4$ satisfies the recurrence $y_n + 4 = 2(y_{n-1} + 4)$, and $y_0 + 4 = 4$, so we have $y_n + 4 = 2^{n+2}$, giving $y_n = 2^{n+2} - 4$.

   I'm not actually sure about z_n.

8. *Prove that the area of a triangle with coordinates (a,b), (c,d), and (e,f) is given by*

$$\frac{1}{2}\left|\det\begin{pmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{pmatrix}\right| = \frac{1}{2}\left|ad + be + cf - af - bc - de\right|.$$

For a particular configuration of points, say $a < c < e$ and $d < b < f$, the area of the triangle is the area of the rectangle containing the triangle, less the extraneous corner pieces, which is

$$(e-a)(f-d) - \frac{1}{2}(c-a)(b-d) - \frac{1}{2}(e-c)(f-d) - \frac{1}{2}(e-a)(f-b).$$

This simplifies to the correct formula.

One could check this formula for all other shapes the points could make. It might help to observe that swapping any pair of points changes neither the value of the formula, nor the area.

9. *(AIME 2005) The points $A = (p,q)$, $B = (12,19)$, and $C = (23,20)$ form a triangle of area 70. The median from A to side BC has slope -5. Find the largest possible value of $p+q$.*

Using the formula above, we get that $11q - p - 197 = \pm 140$. Both cases should be tried, but $11q - p = 197 + 140$ is the one that will work.

The midpoint of BC is $(\frac{35}{2}, \frac{39}{2})$, so we have $q - \frac{39}{2} = -5(p - \frac{35}{2})$ (since A lies on the median from A to BC), which simplifies to $5p + q = 107$.

The two equations for p and q yield $p = 15$ and $q = 32$, so $p + q = 47$. (The other case yields $p = 20$ and $q = 7$, for a sum of only 27.)

10. (a) *Prove that the medians of a triangle can be translated (without rotating the line segments) to form the sides of a new triangle.*

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be the vectors corresponding to the vertices of the triangle. Then the median from the first vertex has vector $\frac{\vec{v}+\vec{w}}{2} - \vec{u} = \frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} - \vec{u}$. Similarly, the other two medians have vectors $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{w} - \vec{v}$ and $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} - \vec{w}$.

These three vectors add to $\vec{0}$, representing the fact that if you start at $\vec{0}$, translate by one median, then the second, then the third, then you get back to where you started: in other words, the medians, without rotation, can be used to form a triangle.

(b) *The medians of $\triangle ABC$ are translated to form the sides of $\triangle DEF$, and the medians of $\triangle DEF$ are translated to form the sides of $\triangle GHI$. Prove that $\triangle ABC$ and $\triangle GHI$ are similar, and compute the coefficient of similarity.*

The three vertices of $\triangle DEF$ might have vectors $\frac{\vec{u}-\vec{v}}{2}$, $\frac{\vec{v}-\vec{w}}{2}$, and $\frac{\vec{w}-\vec{u}}{2}$. (You can check that the pairwise differences agree with what we get above; the formula is flexible since we can translate all three vertices however we like.)

Iterating this, the three vertices of $\triangle GHI$ might have vectors $\frac{1}{2}\vec{u} - \frac{1}{4}\vec{v} - \frac{1}{4}\vec{w}$, $\frac{1}{2}\vec{v} - \frac{1}{4}\vec{u} - \frac{1}{4}\vec{w}$, and $\frac{1}{2}\vec{w} - \frac{1}{4}\vec{u} - \frac{1}{4}\vec{v}$.

Translate this by $\frac{\vec{u}+\vec{v}+\vec{w}}{4}$, and you get vectors $\frac{3}{4}\vec{u}$, $\frac{3}{4}\vec{v}$, and $\frac{3}{4}\vec{w}$: the original triangle, scaled by $\frac{3}{4}$.

11. *Find the equation of the line that bisects the angle formed in the first quadrant by the x-axis and the line $y = mx$.*

One approach is to find the angle bisector of the triangle with vertices $(0,0)$, $(1,0)$, and $(1,m)$. By the angle bisector theorem, it will cut the side $(1,0)$ to $(1,m)$ in the ratio $1 : \sqrt{m^2+1}$, so it will pass through the point $(1, \frac{m}{1+\sqrt{m^2+1}})$ as well as $(0,0)$. Such a line has equation $y = \frac{m}{1+\sqrt{m^2+1}}x$.

12. *(INMO 2009) Let P be a point inside $\triangle ABC$ such that $\angle BPC = 90^\circ$ and $\angle BAP = \angle BCP$. Let M, N be the midpoints of AC, BC respectively. Suppose $BP = 2PM$. Prove that A, P, and N are collinear.*

Choose $P = (0,0)$, $B = (0,b)$, and $C = (c,0)$. We have one degree of freedom to simplify the problem, so let's make $A = (-1,-a)$. Now we solve for $a, b, c > 0$ to meet the conditions of the problem.

Let $\alpha = \angle BAP = \angle BCP$ and β be the angle AP makes with the line $y = -a$. Then $\tan\alpha = \frac{b}{c}$, $\tan\beta = a$, and $\tan(\alpha+\beta) = a+b$. So we have

$$a + b = \frac{a + \frac{b}{c}}{1 - \frac{ab}{c}} = \frac{ac+b}{c-ab}.$$

This simplifies to $b(c-a^2-ab-1) = 0$, so unless $b = 0$ (a trivial case), we have $c-1 = a(a+b)$.

The coordinates of M are $(\frac{c-1}{2}, \frac{a}{2})$, so we also have the constraint $(c-1)^2 + a^2 = b^2$, or $(c-1)^2 = b^2 - a^2 = (b+a)(b-a)$.

Thus $a^2(a+b)^2 = (a+b)(b-a)$; ignoring a few trivial cases (which can be checked separately), we get $b = \frac{a^3+a}{1-a^2}$, and then $c = \frac{a^2+1}{1-a^2}$.

But then the slope of PN is $\frac{b}{c} = a$, the same as the slope of AP. So A, P, and N are collinear.