Combinatorics — Victor Xu

Recursion Solutions

Western PA ARML Practice — *December 13, 2015*

1 Solutions

1. How many ways are there to divide a 1×8 strip into blocks of size 1×1 or 1×2?

   Any $1 \times n$ strip is either a $1 \times (n-1)$ strip with a 1×1 strip with a 1×1 square at the end or a $1 \times (n-2)$ strip with a 1×2 rectangle at the end. Thus, the number of ways to divide a $1 \times n$ strip is equal to the number of ways to divide a $1 \times (n-1)$ strip plus the number of ways to divide a $1 \times (n-2)$ strip, for $n \geq 3$. There is 1 way to divide a 1×1 strip and 2 ways to divide a 1×2 strip. Now, we simply calculate the number of ways to divide a $1 \times k$ strip for $3 \leq k \leq 8$ sequentially to arrive at $\boxed{34}$.

2. How many ways are there to divide a 1×8 strip into blocks of size 1×1 or 1×2 or 1×3?

   This problem is very similar to the first, except a $1 \times n$ strip may also be a $1 \times (n-3)$ strip with a 1×3 rectangle on the end. We use the same process as we did in problem 1, being careful to note that there are 4 ways to divide a 1×3 strip, to get $\boxed{81}$ as the final answer.

3. How many ways are there to cover a 2×5 rectangle with 1×2 dominoes and 1×1 squares? Dominoes may be rotated.

   Here, we can again look at how a 2×5 rectangle can be constructed. Let (a, b) denote a shape with a squares in the top row and b squares in the bottom row. Let's look at how the bottom left cell can be covered. It could be covered by a square, in which case we are left with a $(5, 4)$ piece, or by a domino, which will leave us with a $(4, 4)$ piece or a $(5, 3)$ piece depending on how it is rotated. If we have a non-rectangular piece, we can look at the ways to cover the rightmost piece on the longer row (the square that "sticks out the most"). We can either cover this with a 1×1 square or a 1×2 domino. Using these two recurrences, we can construct a table for the number of ways to construct a rectangle. Each column and row in the table will denote the number of squares in the first row or second row of the piece respectively.

	0	1	2	3	4	5
0	0	1	2	3	5	8
1	1	2	3	5	8	13
2	2	3	7	10	17	27
3	3	5	10	27	37	64
4	5	8	17	37	101	138
5	8	13	27	64	138	377

   So we get $\boxed{377}$ as our final solution.

4. How many 6 character strings with the letters A, B, C are there with the property that there are never two same letters adjacent to each other or with only one letter in between? (For example, CABCAB is acceptable but ABCBBA is not).

   After the first two letters, every letter is the one letter that is not used. There are 3 choices for the first letter and 2 for the second, for a total of $\boxed{6}$ choices. (Initially this question was intended to be a little more difficult, but I made a mistake. Recursion is difficult enough as it is, so (I think) this was not too bad.)

5. (AIME 2001) A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

   Let's look at the last house that receives mail recursively. Let's call $f(n)$ the number of ways to distribute mail to the first n houses, where the nth house recieves mail. Then, $f(n) = f(n-2) + f(n-3)$, since the last house that received mail was either the house 2 spaces ago or the house 3 spaces ago. If we have n houses, then the number of ways to distribute mail to all of the houses is $f(n) + f(n-1) + f(n-2)$, since if the last house that was given mail is the $n-3^{rd}$ or earlier, this would not satisfy the given condition. Now, we can compute $f(n)$ recursively to find the answer, which is $\boxed{351}$.

6. (AMC 12A 2007) Call a set of integers spacy if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

   (A) 121 (B) 123 (C) 125 (D) 127 (E) 129

   We can count the number of valid subsets recursively. Suppose we want to find the number of subsets of $\{1, 2, 3, \ldots, n\}$, that are spacy. Call this number S_n. Either this subset contains n or it doesn't. If it does, then it cannot contain $n-1$ or $n-2$, but we can make any spacy subset with n by adding it to a spacy subset of $\{1, 2, 3, \ldots, n-3\}$,. If the subset doesn't contain n, then it is a subset of $\{1, 2, 3, \ldots, n-1\}$,. Therefore, $S_n = S_{n-1} + S_{n-3}$ for $n \geq 4$. Now, we simply do some computation to find S_12, to obtain $\boxed{129}$.

7. Find a recursive formula for the number of ways to divide n distinguishable objects into m indistiguishable boxes. (These numbers are called Stirling numbers of the second kind).

   I made a mistake with the problem statement; it is somewhat ambiguous. None of the boxes can be empty. I apologize if you worked on this problem and got stuck because of this ambiguity.

   Let $S(n, m)$ be the number of ways to do this. Either the n^{th} object is in its own box or it isn't. If it is, then there are $S(n-1, m-1)$ ways to distribute the other $n-1$ objects among the $m-1$ boxes. If it isn't, then we can look at the number of ways to put $n-1$ objects in m boxes, and multiply this by m since we can choose one of the (now all different) boxes to put the n^{th} object in. Therefore, we find that $S(n, m) = S(n-1, m-1) + kS(n-1, m)$. For completeness, note that $S(n, 1) = 1$ for all n.

8. (http://fivethirtyeight.com/features/whats-the-best-way-to-drop-a-smartphone/) You work for a tech firm developing the newest smartphone that supposedly can survive falls from great heights. Your firm wants to advertise the maximum height from which the phone can be dropped without breaking.

   You are given two of the smartphones and access to a 100-story tower from which you can drop either phone from whatever story you want. If it doesnt break when it falls, you can retrieve it and use it for future drops. But if it breaks, you dont get a replacement phone.

   Using the two phones, what is the minimum number of drops you need to ensure that you can determine exactly the highest story from which a dropped phone does not break? (Assume you know that it breaks when dropped from the very top.) What if, instead, the tower were 1,000 stories high?

   The solution is now posted on 538! Read the solution, and take a look at this week's problem, here: http://fivethirtyeight.com/features/which-geyser-gushes-first/

9. (Putnam 2015) Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|i - j| = 1 \text{ implies } |\pi(i) - \pi(j)| \leq 2$$

   for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

   does not depend on n, and find its value.

   Let C_n denote the number of permutations satisfying the above property with the additional condition that the permutation maps 1 to 1.

   First, we demonstrate that $C_n = C_{n-1} + C_{n-3} + 1$, for $n \geq 4$. We can break up the permutations in C_n into three cases: those that map 2 to 2, those that map 2 to 3, 3 to 2, and 4 to 4, and the one that goes $1, 3, 5, ..., 6, 4, 2$. There are C_{n-1} permutations in the first case and C_{n-3} permutations in the second case since we can ignore the first element and first three elements respectively and the condition for the remaining elements is equivalent to forming a permutation in C_{n-1} or C_{n-3}.

   Next, we show that $P_n = 2(C_n + \sum_{i=1}^{n-2} C_i)$. This can be shown by breaking up the permutations based on what they map 1 to. If 1 is mapped to 1 or n, there are clearly C_n possible permutations. Otherwise, until we map to a complete block of 1 to k or k to n for some k, we must increase by 2 or decrease by 2 in what we map adjacent numbers to since if we did not do this, we would split the remaining numbers into two disconnected halves and it would be impossible to complete the permutation. Therefore, if we map 1 to k, we either map to everything from 1 to k in the next elements, and then have $k+1$ in the permutation, in which case there are C_{n-k} ways to complete the permutation, or we map to everything from k to n in the next elements, and then map to $k-1$ next in the permutation, in which case there are C_{k-1} ways to complete the permutation. Summing up all these cases gives the desired equation.

   Lastly, we substitute our recursive formulae into the expression given in the problem. This is left as an exercise to the reader, but the final solution is $\boxed{4}$.