Optimization Problems

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1. **Triangle-free graphs.**

   A *triangle* in a graph is a set of 3 vertices such that all the edges between them are present.

   (a) Find a graph with 10 vertices and 25 edges that contains no triangles.

   (b) Show that any graph with 10 vertices that contains no triangles can have at most 25 edges.

   (Hint: in general, we can say that any graph with n vertices and no triangle has at most $\frac{n^2}{4}$ edges. You can prove this by induction on n, among other methods.)

2. **Monotonic subsequences.**

   The integers $1, 2, \dots, 100$ are written in some arbitrary order. A *monotonic subsequence* is a subsequence of integers $a_1, a_2, \dots, a_k$ in the order they are written down, such that either $a_1 < a_2 < \cdots < a_k$ or $a_1 > a_2 > \cdots > a_k$.

   (a) Show that you can always find a monotonic subsequence of length 10.

   (b) Find an ordering with no monotonic subsequence of length 11.

   (c) How does this generalize to integers $1, 2, \dots, n$ for arbitrary n?

3. **Coloring points in the grid.**

   How large an $a \times b$ grid of points can you color red and blue without getting a rectangle whose 4 corners have the same color?[1]

   (a) If you haven't done so already, prove that there is always such a rectangle in a coloring of the 3×7 grid – you've seen this problem two weeks ago.

   (b) What coloring of a smaller grid immediately follows from the proof? Can you color a 6×6 grid (that being the largest grid not containing a 3×7 grid)? If not, what are the sizes of grids you can color?

   (c) Generalize the problem to $k > 2$ colors.

[1] We assume the sides of the rectangle are parallel to the sides of the grid to avoid funny business.

4. **Sidon sets.**

A *Sidon set* is a set of integers such that all the sums of two numbers in the set are different: if a, b, c, d are 4 numbers from the set, and $a + b = c + d$, then either $a = c$ and $b = d$, or $a = d$ and $b = c$. The set $\{1, 2, 5, 7\}$ is Sidon; the set $\{1, 2, 5, 6\}$ is not, because $1 + 6 = 2 + 5$.

(a) Use the pigeonhole principle to prove an upper bound on the number of elements in a Sidon set containing only numbers between 1 and 100. (There are two bounds to be obtained here, one better than the other.)

(b) Use any method you like to find as large a Sidon set as possible containing only numbers between 1 and 100.

5. **Coloring complete graphs.**

The *complete graph* K_n is the graph with n vertices in which each pair of vertices is connected by an edge. We color the edges of K_n red and blue.

(a) How large can n get so that such a coloring exists without obtaining a red or blue triangle (i.e. 3 vertices of K_n such that the 3 edges between them are all the same color)?

(b) The *Ramsey number* $R(t)$ is the smallest n such that in any coloring of K_n, there exists a *monochromatic* K_t: t vertices such that all of the edges between them are the same color.

In part (a) we've found $R(3)$. Do your best to find $R(4)$.

(c) We generalize $R(t)$ to $R(s, t)$: the smallest n such that any coloring of K_n has a red K_s or a blue K_t (so $R(t) = R(t, t)$ in this new notation). Prove that $R(s, t) \leq R(s, t - 1) + R(s - 1, t)$, and deduce an upper bound on $R(t)$.

(Hint: use the pigeonhole principle.)

(d) Suppose $n = 2^{t/2}$, and we take a random coloring of K_n: for each edge, we flip a coin to decide if it is red or blue. Count the expected number of monochromatic K_t's: this is the sum, over all sets of t vertices, of the probability that all edges between them are the same color.

Show that the answer is less than 1. This tells us that a coloring of K_n with no monochromatic K_t's exists: if it didn't, then there would always be at least one monochromatic K_t, and the expected number would be at least 1 as well. So $R(t) > 2^{t/2}$.