

The Basel Problem

Numerous Proofs

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April 11, 2013

Abstract

The **Basel Problem** was first posed in 1644 and remained open for 90 years, until Euler made his first waves in the mathematical community by solving it. During his life, he would present three different solutions to the problem, which asks for an evaluation of the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Since then, people have continually looked for new, interesting, and enlightening approaches to this same problem. Here, we present 5 different solutions, drawing from such diverse areas as complex analysis, calculus, probability, and Hilbert space theory. Along the way, we'll give some indication of the problem's intrigue and applicability.

- 0 Introduction
 - History
 - Intrigue
- 1 Proof: $\sin x$ and L'Hôpital
- 2 Proof: $\sin x$ and Maclaurin
- Analysis: $\sin x$ as an infinite product
- 3 Proof: Integral on $[0, 1]^2$
- 4 Proof: $L^2[0, 1]$ and Parseval
- 5 Proof: Probability Densities
- 6 References

Pietro Mengoli

- Italian mathematician and clergyman (1626–1686) [5]
- PhDs in math and civil/canon law
- Assumed math chair at Bologna after adviser Cavalieri died
- Priest in the parish of Santa Maria Maddelena in Bologna
- Known (nowadays) for work in infinite series:
Proved: harmonic series diverges, alternating harmonic series sums to $\ln 2$, Wallis' product for π is correct
- Developed many results in limits and sums that laid groundwork for Newton/Leibniz
- Wrote in “abstruse Latin”; Leibniz was influenced by him [6]

Problem Statement & Early Work

Mengoli posed the **Basel Problem** in 1644.

Find the numerical value of:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

- Wallis, 1655: “I know it to 3 decimal places.”
- Jakob Bernoulli, 1689: “It’s less than 2. Help us out!”
- Johann/Daniel Bernoulli, 1721: “It’s about $\frac{8}{5}$.”
- Goldbach, 1721ish: “It’s between $\frac{41}{35} = 1.64$ and $\frac{5}{3} = 1.\bar{6}$.”
- Leibniz, DeMoivre: “ ??? ”

Part of the difficulty is that the series converges *slowly*:

$n = 10^2 \Rightarrow 1$ place, $n = 10^3 \Rightarrow 2$ places, $n = 10^5 \Rightarrow 4$ places,

$n = 10^6 \Rightarrow 5$ places

Euler Emerges!

- Since this had stumped so many brilliant minds, Euler's solution in 1735 (at age 28) brought him immediate fame.
- He was born in Basel. (Problem name comes from publishing location of Jakob Bernoulli's *Tractatus de seriebus infinitis*, though.)
- Studied under Johann Bernoulli, starting 1721.
- Was working on it by 1728, calculating partial sums.
- Published a more rigorous proof in 1741, and a third in 1755
- His techniques inspired Weierstrass (to rigorize his methods and develop analysis) and Riemann (to develop the zeta function and the Prime Number Theorem) in the 1800s

Riemann ζ function: definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \quad \text{for } \Re(s) > 1$$

Non-series definition:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

is the analytic extension of the factorial function to \mathbb{C} .

Notice $\forall k \in \mathbb{N}$. $\zeta(-2k) = 0$; these are *trivial* roots.

Riemann hypothesis: $\Re z = \frac{1}{2}$ for every nontrivial root.

Riemann ζ function: applications

If we “knew” $\zeta(s)$, we could evaluate

$$f(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log(t)}$$

where $\text{Li}(x) = \int_0^{\infty} \frac{dx}{\log(x)}$ and \sum_{ρ} is over *nontrivial roots* of $\zeta(s)$.

This would help us find the primes function $\pi(x)$ by

$$\pi(x) = f(x) - \frac{1}{2}f\left(x^{1/2}\right) - \frac{1}{3}f\left(x^{1/3}\right) - \dots$$

Riemann was likely motivated by **Prime Number Theorem**:

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n / \log n} = 1$$

Euler's work on ζ

Euler's 1741 proof actually adapted his earlier method to find

$$\forall k \in \mathbb{N}. \zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}}{2(2k)!} B_{2k}$$

where B_{2k} is the **Bernoulli Number**, defined by

$$B_m = 1 - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$$

with $B_0 = 1$ and $B_1 = -\frac{1}{2}$.

Interestingly, not much is known about $\zeta(2k+1)$.

Closed form for $\zeta(3)$ is open! (It's irrational [1], proven 1979.)

Proof 1: History

- L'Hôpital published *Analyse des infiniment petits* in 1696. Standard calc text for many years, until ...
- Euler published *Institutiones calculi differentialis* in 1755. New standard! [7]
- Euler's book includes a discussion of indeterminate forms.
- He gave no credit to L'Hôpital (who, in turn, gave no credit to Johann Bernoulli), but he did state and prove the rule, and provide several striking examples, including

$$\lim_{x \rightarrow 1} \frac{x^x - x}{1 - x + \ln x} = -2 \quad \text{and} \quad \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

and ...

Proof 1: Sketch

1. Write $\sin x$ as an infinite product of linear factors
(More on this later ...)
2. Take log of both sides to get a sum
3. Differentiate
4. Make a surprising change of variables
5. Plug in $x = 0$ and use L'Hôpital thrice
6. Sit back and smile smugly at your brilliance

Proof 1: Details

$$\sin t = t \left(1 - \frac{t}{\pi}\right) \left(1 + \frac{t}{\pi}\right) \left(1 - \frac{t}{2\pi}\right) \left(1 + \frac{t}{2\pi}\right) \cdots$$

since $\sin t$ has roots precisely at $t \in \mathbb{Z}$

$$\sin(\pi y) = \pi y (1 - y) (1 + y) \left(\frac{2 - y}{2}\right) \left(\frac{2 + y}{2}\right) \cdots$$

$$= \pi y (1 - y^2) \left(\frac{4 - y^2}{4}\right) \left(\frac{9 - y^2}{9}\right) \cdots$$

$$\ln(\sin(\pi y)) = \ln \pi + \ln y + \ln(1 - y^2) + \ln(4 - y^2) - \ln 4 + \cdots$$

Differentiate with respect to $y \dots$

Proof 1: Details

$$\ln(\sin(\pi y)) = \ln \pi + \ln y + \ln(1 - y^2) + \ln(4 - y^2) - \ln 4 + \dots$$

$$\frac{\pi \cos(\pi y)}{\sin(\pi y)} = \frac{1}{y} - \frac{2y}{1 - y^2} - \frac{2y}{4 - y^2} - \frac{2y}{9 - y^2} - \dots$$

$$\frac{1}{y} + \frac{1}{1 - y^2} + \frac{1}{4 - y^2} + \frac{1}{9 - y^2} + \dots = \frac{1}{2y^2} - \frac{\pi \cos(\pi y)}{2y \sin(\pi y)}$$

$$\text{COV: } y = -ix \iff y^2 = -x^2$$

$$\frac{1}{1 + x^2} + \frac{1}{4 + x^2} + \frac{1}{9 + x^2} + \dots = -\frac{1}{2x^2} + \frac{\pi \cos(-i\pi x)}{2ix \sin(-i\pi x)}$$

Proof 1: Details

Use **Euler's Formula** (he's everywhere!) to find:

$$\frac{\cos(z)}{\sin(z)} = \frac{\frac{1}{2}(e^{iz} + e^{-iz})}{\frac{1}{2i}(e^{iz} - e^{-iz})} = \frac{i(e^{2iz} + 1)}{e^{2iz} - 1}$$

$$\frac{\pi \cos(-i\pi x)}{2ix \sin(-i\pi x)} = \frac{\pi}{2ix} \cdot \frac{i(e^{2\pi x} + 1)}{e^{2\pi x} - 1} = \frac{\pi}{2x} \cdot \frac{(e^{2\pi x} - 1) + 2}{e^{2\pi x} - 1}$$

$$= \frac{\pi}{2x} + \frac{\pi}{x(e^{2\pi x} - 1)}$$

Substitute this back into the previous equation ...

Proof 1: Details

$$\begin{aligned}
 \frac{1}{1+x^2} + \frac{1}{4+x^2} + \frac{1}{9+x^2} + \cdots &= -\frac{1}{2x^2} + \overbrace{\frac{\frac{\pi}{2x} + \frac{\pi}{x(e^{2\pi x} - 1)}}{\pi \cos(-i\pi x)}} \\
 &= \frac{\pi x - 1}{2x^2} + \frac{\pi}{x(e^{2\pi x} - 1)} \\
 &= \frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2x^2 e^{2\pi x} - 2x^2}
 \end{aligned}$$

Plug in $x = 0$:

LHS is the desired sum, $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

RHS is $\frac{0}{0}$. L'Hôpital to the rescue!

Proof 1: Details

$$\begin{aligned}
 \frac{\pi x e^{2\pi x} - e^{2\pi x} + \pi x + 1}{2x^2 e^{2\pi x} - 2x^2} &\mapsto \frac{\pi - \pi e^{2\pi x} + 2\pi^2 x e^{2\pi x}}{4x e^{2\pi x} + 4\pi x^2 e^{2\pi x} - 4x} \\
 &\mapsto \frac{\pi^3 x e^{2\pi x}}{e^{2\pi x} + 4\pi x e^{2\pi x} + 2\pi^2 x^2 e^{2\pi x} - 1} \\
 &\mapsto \frac{\pi^3}{4\pi + 4\pi^2 x + 2\pi e^{-2\pi x}} \\
 &\mapsto \frac{\pi^3}{4\pi + 2\pi} \\
 &\mapsto \frac{\pi^2}{6}
 \end{aligned}$$

Everything works! Because Euler said so. And it does. □

Proof 1: Summary

1. Write $\sin x$ as an infinite product of linear factors
2. Take log of both sides to get a sum
3. Differentiate
4. Make a surprising change of variables
5. Plug in $x = 0$ and use L'Hôpital thrice
6. ~~Sit back and smile smugly at your brilliance~~ Cite Euler

“What a marvelous derivation it was. It boasted an all-star cast of transcendental functions: sines, cosines, logs, and exponentials. It ranged from the real to the complex and back again. It featured L'Hôpital's Rule in a starring role. Of course, none of this would have happened without the fluid imagination of Leonhard Euler, symbol manipulator extraordinaire.”

–William Dunham [7]

Proof 2: History & Sketch

- Euler's (and the world's) first proof, from 1735.
 - One of the easiest methods to remember.
1. Write $\sin x$ as an infinite product of linear factors
(More on this later ...)
 2. Also find the Maclaurin series for $\sin x$
 3. Compare the coefficients of x^3
 4. Marvel at the coincidence

Proof 2: Details

$$\begin{aligned}
 \sin(\pi x) &= \pi x (1 - x^2) \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \cdots \\
 &= \pi x \\
 &\quad + \pi x^3 \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right] \\
 &\quad + \pi x^5 \left[\frac{1}{1 \cdot 4} + \frac{1}{1 \cdot 9} + \cdots + \frac{1}{4 \cdot 9} + \cdots\right] \\
 &\quad + \cdots \\
 \sin(\pi x) &= \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots \\
 &= \pi x - \frac{\pi^3}{6} x^3 + \frac{\pi^5}{120} x^5 - \cdots
 \end{aligned}$$

Proof 2: Details

$$\begin{aligned}
 \sin(\pi x) &= \pi x (1 - x^2) \left(1 - \frac{x^2}{4}\right) \left(1 - \frac{x^2}{9}\right) \cdots \\
 &= \pi x \\
 &\quad - \pi x^3 \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right] \\
 &\quad + \pi x^5 \left[\frac{1}{1 \cdot 4} + \frac{1}{1 \cdot 9} + \cdots + \frac{1}{4 \cdot 9} + \cdots\right] \\
 &\quad - \cdots
 \end{aligned}$$

$$\begin{aligned}
 \sin(\pi x) &= \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots \\
 &= \pi x - \frac{\pi^3}{6} x^3 + \frac{\pi^5}{120} x^5 - \cdots
 \end{aligned}$$

Proof 2: Details

Thus,

$$\pi \left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \right] = \frac{\pi^3}{6}$$



Proof 2: Details

Thus,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$



Proof 2: Summary

1. Write $\sin x$ as an infinite product of linear factors
2. Also find the Maclaurin series for $\sin x$
3. Compare the coefficients of x^3

Not particularly *challenging*, so to speak. You could present this to a Calc II class and be convincing!

Hints at the deeper relationships between products/series.

The actual validity depends heavily on complex analysis, and would only be officially resolved in the mid 1800s by Weierstrass.

Main Question: Why can we “factor” $\sin x$ into linear terms?

Analysis: $\sin x$ as an infinite product

Weierstrass Factorization Theorem

Let f be an entire function and let $\{a_n\}$ be the nonzero zeros of f repeated according to multiplicity. Suppose f has a zero at $z = 0$ of order $m \geq 0$ (where order 0 means $f(0) \neq 0$). Then $\exists g$ an entire function and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m \exp(g(z)) \prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{a_n} \right)$$

where

$$E_n(y) = \begin{cases} (1 - y) & \text{if } n = 0, \\ (1 - y) \exp \left(\frac{y^1}{1} + \frac{y^2}{2} + \dots + \frac{y^n}{n} \right) & \text{if } n = 1, 2, \dots \end{cases}$$

This is a direct generalization of the Fundamental Theorem of Algebra. It turns out that for $\sin(\pi x)$, the sequence $p_n = 1$ and the function $g(z) = \log(\pi z)$ works.



Proof 3: History

- Published by Tom Apostol in 1983 in *Mathematical Intelligencer*. Two page proof, with profile pic:



- Cites Apéry's 1979 proof that $\zeta(2), \zeta(3) \notin \mathbb{Q}$. Specifically, a shorter proof by Beukers [4] uses integrals.
- "This evaluation has been presented by the author for a number of years in elementary calculus courses, but does not seem to be recorded in the literature."
- Simple, clear exposition, but no indication of *insight*.

Proof 3: Sketch

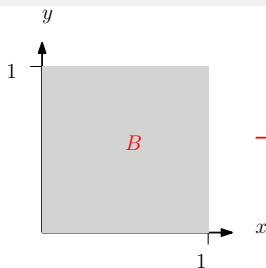
1. Consider $\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy$
2. Observe it evaluates to $\zeta(2)$ by writing fraction as infinite series and evaluating term by term
3. Evaluate another way by rotating $[0, 1]^2$ clockwise 45°
4. Make some trig subs, but nothing advanced
5. Applaud this kind of ingenuity 250 years after the fact

Proof 3: Details

Since $\frac{1}{1-r} = 1 + r + r^2 + \dots$ for $|r| < 1$

$$\begin{aligned}\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy &= \int_0^1 \int_0^1 \sum_{n \geq 0} (xy)^n dx dy \\ &= \sum_{n \geq 0} \int_0^1 \int_0^1 x^n y^n dx dy \\ &= \sum_{n \geq 0} \int_0^1 \frac{1}{n+1} [x^{n+1}]_0^1 \cdot y^n dy \\ &= \sum_{n \geq 0} \frac{1}{(n+1)^2} = \sum_{n \geq 1} \frac{1}{n^2}\end{aligned}$$

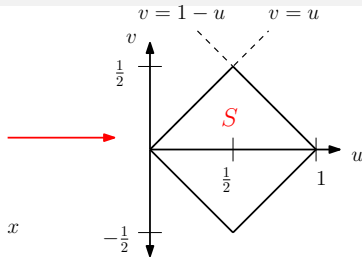
Proof 3: Details



$$u = \frac{x+y}{2}$$

$$x = u - v$$

$$\frac{1}{1-xy} = \frac{1}{1-u^2+v^2}$$

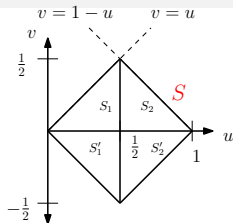


$$v = \frac{y-x}{2}$$

$$y = u + v$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

Proof 3: Details



$$\begin{aligned}
 \iint_B \frac{1}{1-xy} dA &= 2 \iint_S \frac{1}{1-u^2+v^2} dA \\
 &= 4 \int_0^{1/2} \int_0^u \frac{1}{1-u^2+v^2} dv du \quad \left. \vphantom{\int_0^{1/2}} \right\} I_1 \\
 &\quad + 4 \int_{1/2}^1 \int_0^{1-u} \frac{1}{1-u^2+v^2} dv du \quad \left. \vphantom{\int_{1/2}^1} \right\} I_2
 \end{aligned}$$

Proof 3: Details

In general,

$$\int_0^z \frac{dt}{a^2 + t^2} = \left[\frac{1}{a} \tan^{-1} \left(\frac{t}{a} \right) \right]_0^z = \frac{1}{a} \tan^{-1} \left(\frac{z}{a} \right)$$

Use this in I_1 , where $a^2 = (1 - u^2)$ and $z = u$

Use this in I_2 , where $a^2 = (1 - u^2)$ and $z = 1 - u$

$$I_1 = 4 \int_0^{1/2} \tan^{-1} \left(\frac{u}{\sqrt{1 - u^2}} \right) \cdot \frac{du}{\sqrt{1 - u^2}}$$

$$I_2 = 4 \int_{1/2}^1 \tan^{-1} \left(\frac{1 - u}{\sqrt{1 - u^2}} \right) \cdot \frac{du}{\sqrt{1 - u^2}}$$

Proof 3: Details

Evaluating I_1 : Let $u = \sin \theta \iff du = \cos \theta d\theta$

$$\begin{aligned} I_1 &= 4 \int_0^{1/2} \tan^{-1} \left(\frac{u}{\sqrt{1-u^2}} \right) \cdot \frac{du}{\sqrt{1-u^2}} \\ &= 4 \int_0^{\pi/6} \tan^{-1} \left(\frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} \right) \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}} \\ &= 4 \int_0^{\pi/6} \tan^{-1} (\tan \theta) d\theta \\ &= 4 \int_0^{\pi/6} \theta d\theta = 4 \cdot \frac{1}{2} \left(\frac{\pi}{6} \right)^2 = \frac{\pi^2}{18} \end{aligned}$$

Proof 3: Details

Evaluating I_2 : Let $u = \cos(2\theta) \iff du = 2 \sin(2\theta) d\theta$

$$\begin{aligned}
 I_2 &= 4 \int_{1/2}^1 \tan^{-1} \left(\frac{1-u}{\sqrt{1-u^2}} \right) \cdot \frac{du}{\sqrt{1-u^2}} \\
 &= 4 \int_0^{\pi/6} \tan^{-1} \left(\frac{1-\cos(2\theta)}{\sqrt{1-\cos^2(2\theta)}} \right) \frac{2 \sin(2\theta) d\theta}{\sqrt{1-\cos^2(2\theta)}} \\
 &= 8 \int_0^{\pi/6} \tan^{-1} \sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}} d\theta \\
 &= 8 \int_0^{\pi/6} \tan^{-1} \sqrt{\frac{2 \sin^2 \theta}{2 \cos^2 \theta}} d\theta = \frac{\pi^2}{9}
 \end{aligned}$$

Proof 3: Details

Evaluating $I_1 + I_2$:

$$I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$



Proof 3: Summary

1. Consider $\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy$
2. Observe it evaluates to $\zeta(2)$ by writing fraction as infinite series and evaluating term by term
3. Evaluate another way by rotating $[0, 1]^2$ clockwise 45°
4. Make some trig subs ~~but nothing advanced~~

Can be modified using different transformations:

One converts B to a triangle using \tan^{-1} in the uv -transform

Hints at relationship to geometry and trigonometry

Who comes up with these?!

Proof 4: History

- “Textbook proof” in Fourier analysis [9]
- Don’t know first appearance or attribution
- Depends on **Parseval’s Theorem**
- Parseval (1755-1836) was a French analyst, a “shadowy figure” in math history; never elected to Académie des Sciences but proposed 5 times; only 5 papers

Proof 4: Sketch

1. Consider the the space $L^2[0, 1]$ (\mathbb{C} -valued functions)
2. Take complete orthonormal set of exp functions
3. Use $f(x) = x$ and evaluate $\langle f, f \rangle$ as an integral
4. Apply Parseval's Theorem to find $\langle f, f \rangle$ as a sum

Proof 4: Details

Consider the space

$$L^2[0, 1] = \left\{ f : [0, 1] \rightarrow \mathbb{C} \mid \int_0^1 |f|^2 < +\infty \right\}$$

with inner product

$$\langle f, g \rangle = \int_0^1 f \bar{g}$$

This is a nice Hilbert space.

Proof 4: Details

Consider the set of functions

$$\mathcal{S} = \{e_n(x) := \exp(2\pi i n x) \mid n \in \mathbb{Z}\}$$

Claim: \mathcal{S} is an orthonormal basis for $L^2[0, 1]$

Proof: WWTS $\langle e_m, e_n \rangle = 0$ when $m \neq n$ and $= 1$ when $m = n$

$$\langle e_m, e_n \rangle = \int_0^1 \exp(2\pi i m x) \overline{\exp(2\pi i n x)} dx$$

(since $\cos(2\pi n x) - i \sin(2\pi n x) = \cos(-2\pi n x) + i \sin(-2\pi n x)$)

$$= \int_0^1 \exp(2\pi i(m - n)x) dx = \frac{(1 + 0) - 1}{2\pi i(m - n)} = 0$$

unless $m = n$, in which case $\int_0^1 \exp(0) dx = 1$.

Proof 4: Details

Parseval's Theorem:

$$\forall f \in L^2[0, 1]. \quad \langle f, f \rangle = \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2$$

More generally:

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |F(t)|^2 dt$$

where $F(t)$ is the *Fourier transform* of $f(x)$.

Proof uses Fourier Inversion Formula and is straightforward.

Parseval was working on Fourier series.

Proof 4: Details

Let's use $f(x) = x$. (Why not?)

$$\langle f, f \rangle = \int_0^1 x \bar{x} dx = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\langle f, e_0 \rangle = \int_0^1 x \overline{\exp(0)} dx = \int_0^1 x = \frac{1}{2}$$

$$\langle f, e_n \rangle = \int_0^1 x \exp(-2\pi inx) dx \quad \text{for } n \neq 0$$

$$= -\frac{1}{2\pi in} \left([x \exp(-2\pi inx)]_0^1 - \int_0^1 \exp(-2\pi inx) dx \right) \quad \text{IBP}$$

$$= -\frac{1}{2\pi in} \left((1 - 0) + \frac{1}{2\pi in} [\exp(-2\pi inx)]_0^1 \right) = -\frac{1}{2\pi in}$$

Proof 4: Details

Applying Parseval's Theorem:

$$\begin{aligned}\frac{1}{3} = \langle f, f \rangle &= \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2 = \left(\frac{1}{2}\right)^2 + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left| -\frac{1}{2\pi i n} \right|^2 \\ &= \frac{1}{4} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{4\pi^2 n^2} \\ \frac{1}{3} - \frac{1}{4} &= 2 \sum_{n \in \mathbb{N}} \frac{1}{4\pi^2} \cdot \frac{1}{n^2} \\ \frac{\pi^2}{6} &= \sum_{n \in \mathbb{N}} \frac{1}{n^2}\end{aligned}$$

Proof 4: Summary

1. Consider the the space $L^2[0, 1]$ (\mathbb{C} -valued functions)
2. Take complete orthonormal set, $\exp(2\pi inx)$ for $n \in \mathbb{Z}$
3. Use $f(x) = x$ and evaluate $\langle f, f \rangle$ directly
4. Apply Parseval's Theorem to find $\langle f, f \rangle$ as a sum
5. Everything works out

Euler manages to stick his head in. (Euler's Formula!)

Still somehow rooted in his original proofs using series.

Play around with different functions to get related results, e.g.

$$f(x) = \chi_{[0,1/2]} \implies \sum_{n \in \mathbb{N}} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Proof 5: History

- Published in Amer. Math Monthly, August 2011
- Due to Luigi Pace, Dept of Econ & Stats at Udine, Italy [8]
- Inspired by a 2003 note that solves the problem using a double integral on \mathbb{R}_+^2 via Fubini's Theorem
- Probabilists are still studying relationship between independent Cauchy variables and $\zeta(2k)$

Proof 5: Sketch

1. Take a quotient of two random variables $Y = \frac{X_1}{X_2}$
2. Suppose X_1, X_2 are i.i.d. half-Cauchy distributions
3. Find $\Pr(0 \leq Y \leq 1)$ by inspection
4. Find $\Pr(0 \leq Y \leq 1)$ by joint density
5. Turn fraction into series, integrate term by term
6. Smirk mildly at the no longer surprising appearances of $\zeta(2)$

Proof 5: Details

Take $X_1, X_2 : \mathbb{R} \rightarrow \mathbb{R}_+$ random variables. Each is governed by a *probability density function* $p_{X_i} : \mathbb{R}_+ \rightarrow [0, 1]$ satisfying

$$\Pr(a \leq X_i \leq b) = \int_a^b p_{X_i}(t) dt$$

Define $Y = \frac{X_1}{X_2}$.

Claim: Probability density function for Y is

$$p_Y(u) = \int_0^\infty t p_{X_1}(tu) p_{X_2}(t) dt$$

Proof 5: Details

Proof: Appeal to joint density, then simplify:

$$\begin{aligned}\Pr(a \leq Y \leq b) &= \int_0^\infty \int_{at_2}^{bt_2} p_{X_1}(t_1) p_{X_2}(t_2) dt_1 dt_2 \\ &= \int_0^\infty \int_a^b t_2 p_{X_1}(t_2 u) p_{X_2}(t_2) du dt_2 \\ &= \int_a^b \int_0^\infty t_2 p_{X_1}(t_2 u) p_{X_2}(t_2) dt_2 du\end{aligned}$$

“Throw away” the outer integral.

Proof 5: Details

Assign half-Cauchy distribution to X_1, X_2 independently:

$$p_{X_i}(t) = \frac{2}{\pi(1+t)^2}$$

Use this in the formula for $p_Y(u)$ obtained above:

$$\begin{aligned} p_Y(u) &= \frac{4}{\pi^2} \int_0^\infty t \cdot \frac{1}{1+t^2u^2} \cdot \frac{1}{1+t^2} dt \\ &= \frac{2}{\pi^2(u^2-1)} \left[\ln \left(\frac{1+t^2u^2}{1+t^2} \right) \right]_{t=0}^{t=\infty} \\ &= \frac{2}{\pi^2} \cdot \frac{\ln(u^2)}{u^2-1} = \frac{4}{\pi^2} \cdot \frac{\ln u}{u^2-1} \end{aligned}$$

Proof 5: Details

Integrate this expression from 0 to 1:

$$\Pr(0 \leq Y \leq 1) = \int_0^1 p_Y(u) du = \int_0^1 \frac{4}{\pi^2} \cdot \frac{\ln u}{u^2 - 1} du$$

but $\Pr(0 \leq Y \leq 1) = \frac{1}{2}$, obviously, so

$$\int_0^1 \frac{\ln u}{u^2 - 1} du = \frac{\pi^2}{8}$$

Proof 5: Details

Simplify the integral by using $\frac{1}{1-u^2} = 1 + u^2 + u^4 + \dots$:

$$\frac{\pi^2}{8} = \int_0^1 -\frac{\ln u}{1-u^2} du = \sum_{n=0}^{\infty} \int_0^1 \frac{\ln u}{u^{2n}} du = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

This is equivalent to the original claim. (Prove it!) □

Proof 5: Summary

1. Take a quotient of two random variables $Y = \frac{X_1}{X_2}$
2. Suppose X_1, X_2 are i.i.d. half-Cauchy distributions
3. Find $\Pr(0 \leq Y \leq 1)$ by inspection
4. Find $\Pr(0 \leq Y \leq 1)$ by joint density
5. Turn fraction into series, integrate term by term

Similar “tricks” reappearing: e.g. term-by-term integration

Surprising use of probability, but again no indication of insight.

Can probably be modified to find similar results.

Shows this is still an “active” problem, 250+ years later.

References I



W. Dunham

Euler: The Master Of Us All.

MAA, 1999



R. Apéry

Irrationalité de $\zeta(2)$ et $\zeta(3)$

Astérisque, **61** (1979) 11-13



T. Apostol

A Proof that Euler Missed: Evaluating $\zeta(2)$ the Easy Way

Mathematical Intelligencer, **5** (1983) 59-60



F. Beukers

A note on the irrationality of $\zeta(2)$ and $\zeta(3)$

Bull. Lon. Math. Soc., **11** (1979) 268-272



Wikipedia

Pietro Mengoli

http://en.wikipedia.org/wiki/Pietro_Mengoli



MacTutor History of Mathematics

Pietro Mengoli

<http://www-history.mcs.st-andrews.ac.uk/Biographies/Mengoli.html>

References II



W. Dunham

When Euler Met l'Hôpital

Mathematics Magazine, **82** (2009) 16-25



L. Pace

Probabilistically Proving that $\zeta(2) = \pi^2/6$

Amer. Math Monthly, **118** (2011) 641-643



R. Chapman

Evaluating $\zeta(2)$

<http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf>

THANK YOU

