

PHASE TRANSITIONS FOR THIN FILMS

BERNARDO GALVÃO-SOUZA
CARNEGIE MELLON UNIVERSITY

JOINT WORK WITH VINCENT MILLOT



HISTORIC CONTEXT

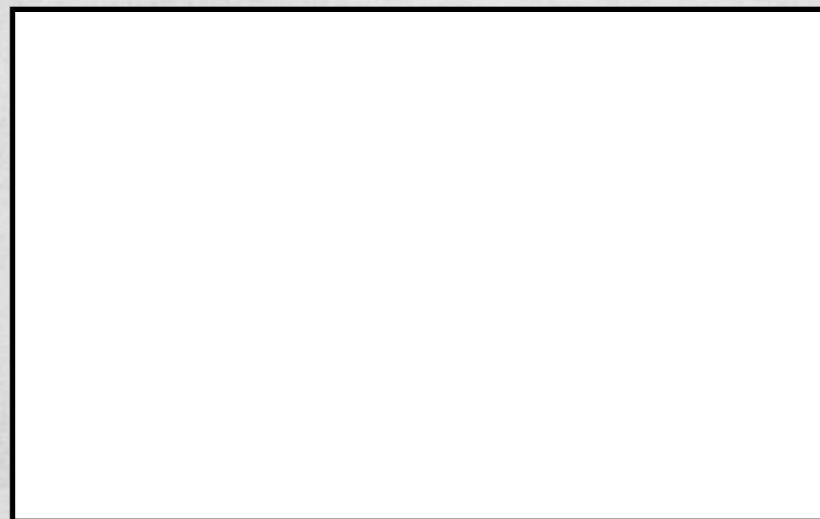
PHASE TRANSITIONS

classical theory for phase transitions for non-interacting fluid

HISTORIC CONTEXT

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classical theory for phase transitions for non-interacting fluid



Ω container

HISTORIC CONTEXT

PHASE TRANSITIONS

classical theory for phase transitions for non-interacting fluid

Ω container

u fluid density satisfies

$$\text{mass: } \int_{\Omega} u \, dx = V$$

HISTORIC CONTEXT

PHASE TRANSITIONS

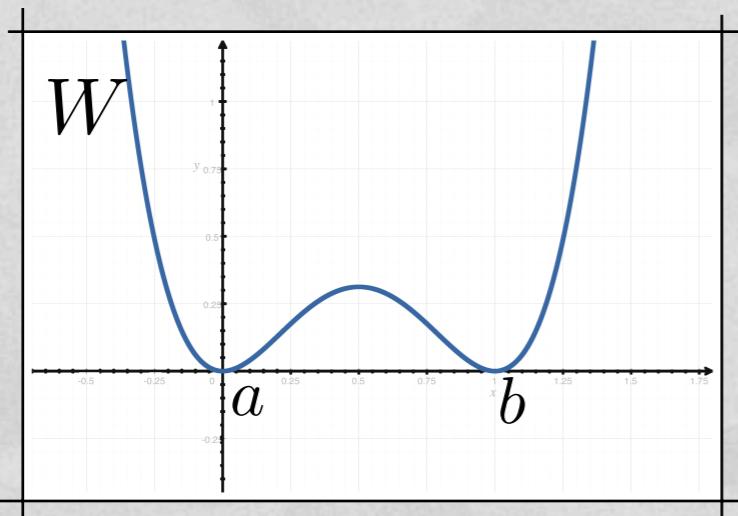
classical theory for phase transitions for non-interacting fluid



Ω container

u fluid density satisfies

$$\min_{\substack{u \in L^1(\Omega) \\ \int_{\Omega} u \, dx = V}} \int_{\Omega} W(u) \, dx$$
$$a|\Omega| < V < b|\Omega|$$

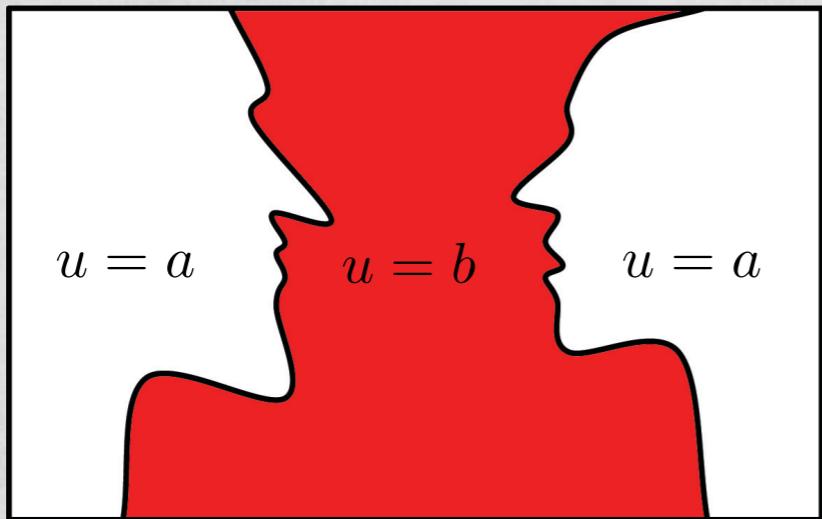


W = Gibbs free energy density

HISTORIC CONTEXT

PHASE TRANSITIONS

classical theory for phase transitions for non-interacting fluid

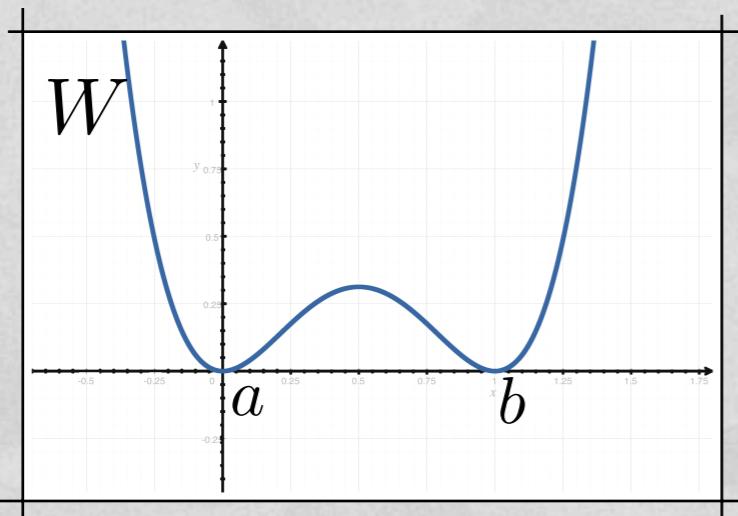


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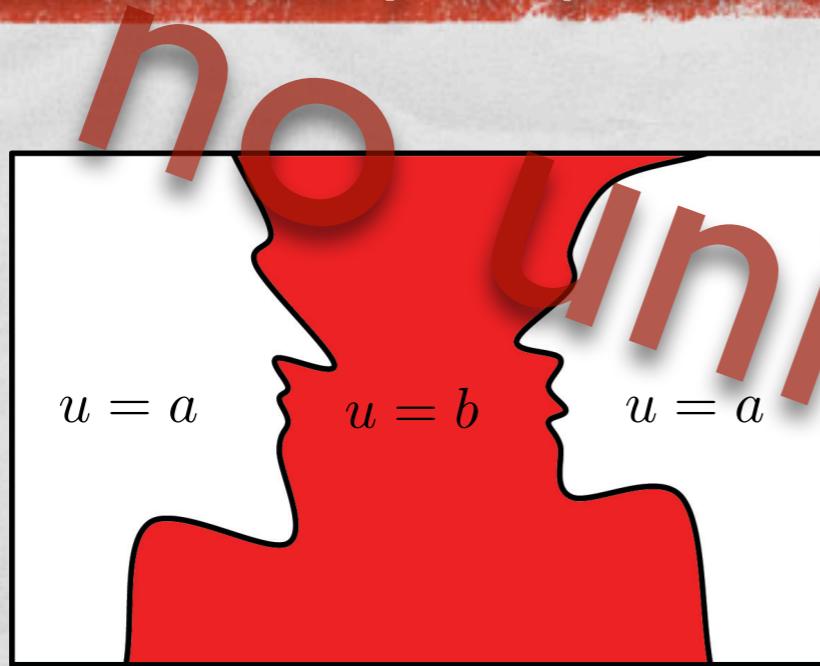


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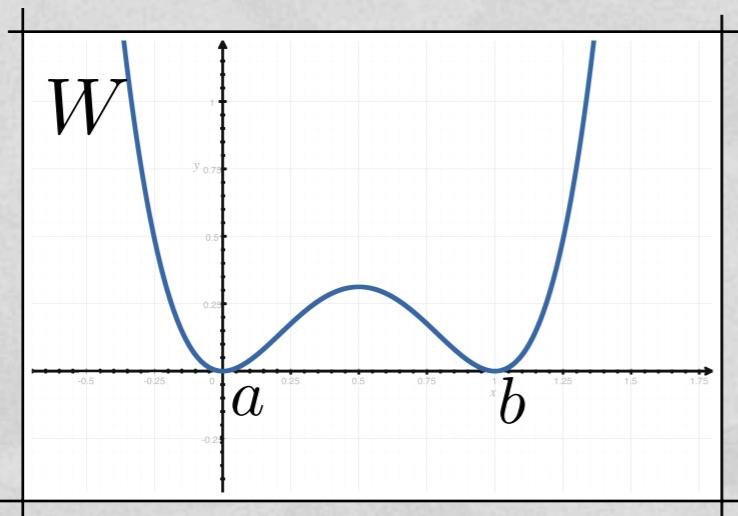
classical theory for phase transitions for non-interacting fluid



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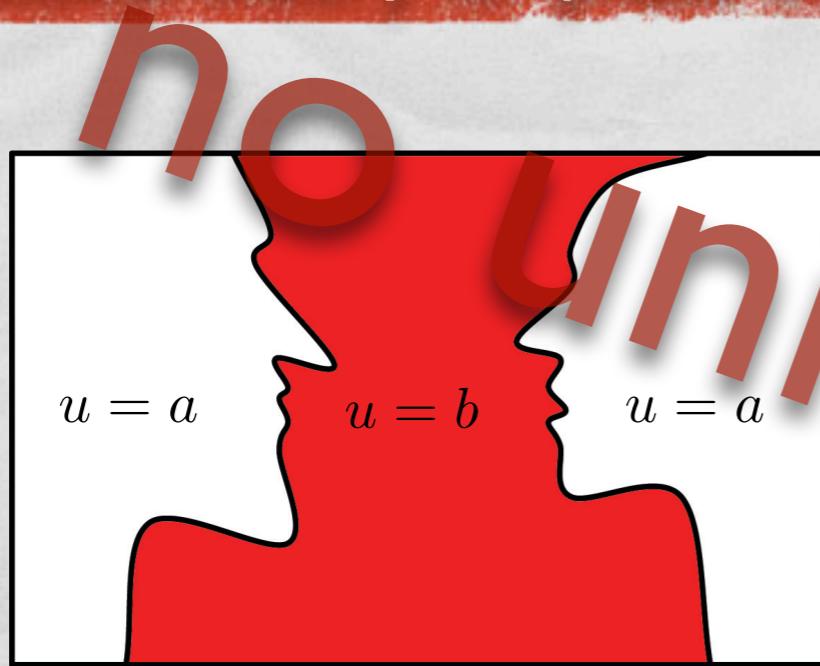


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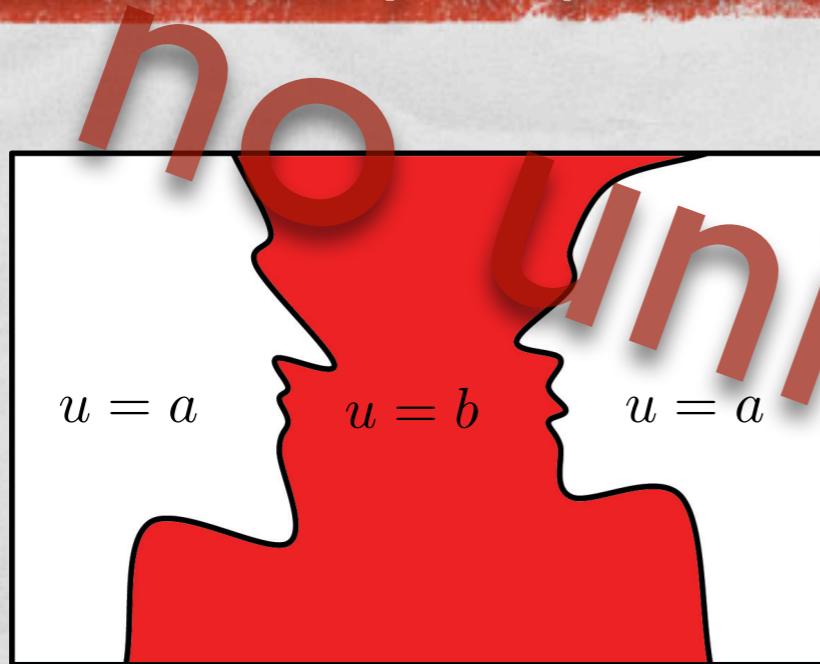
$$\min_{\substack{u \in L^1(\Omega) \\ \int_{\Omega} u \, dx = V}} \int_{\Omega} W(u) \, dx$$
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no uniqueness

HISTORIC CONTEXT

PHASE
TRANSITIONS

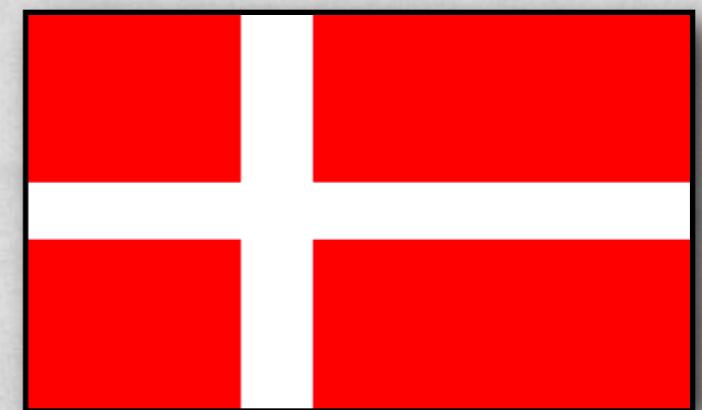
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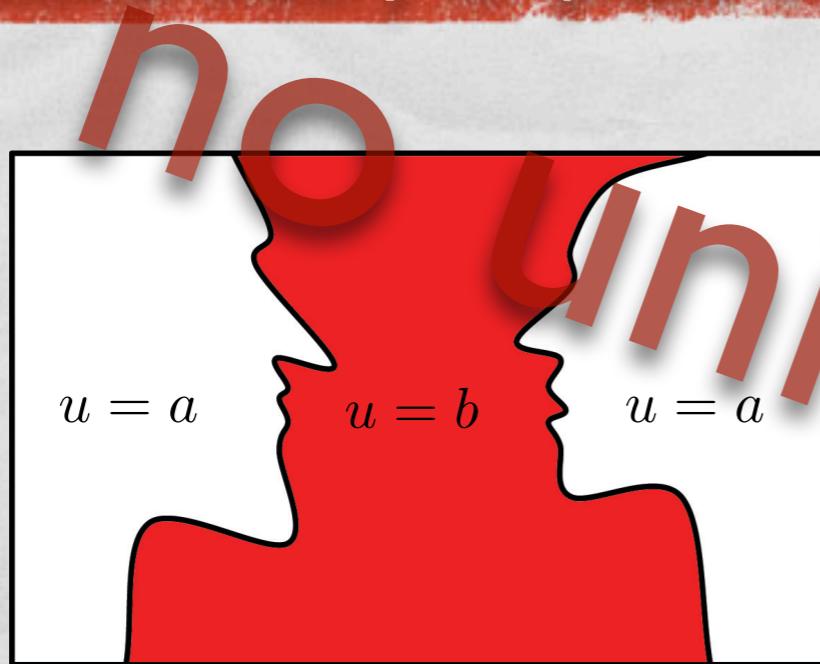
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Van der Waals-Cahn-Hilliard

+ perturbation \Rightarrow selection criterium

$$\begin{aligned} & \min_{\substack{u \in L^1(\Omega) \\ \int_{\Omega} u \, dx = V}} \left\{ \int_{\Omega} W(u) \, dx + \varepsilon^{2k} \int_{\Omega} |D^k u|^2 \, dx \right\} \end{aligned}$$

HISTORIC CONTEXT

PHASE TRANSITIONS

Gurtin '85

$$\varepsilon \int_{\Omega} |Du|^2 dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) dx$$

minimizer u_ε

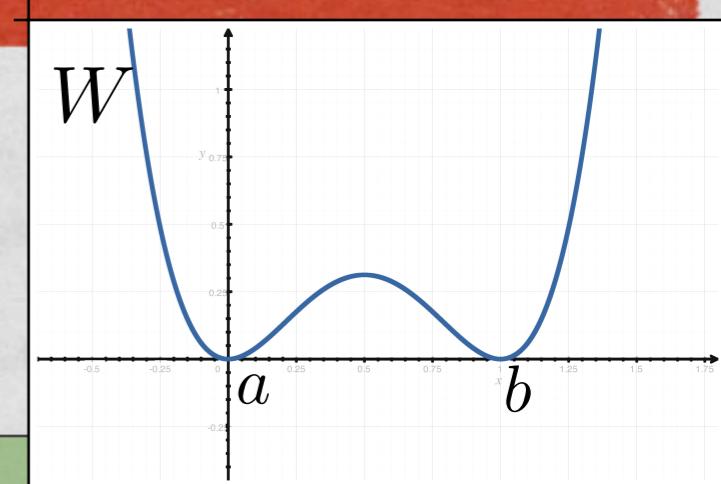
$$u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0^+} u_0$$

u_0 physically-preferred solution

minimizes interface area

$$\text{Per}_\Omega(\{u_0 = a\}) = \min \text{Per}_\Omega(\{u = a\})$$

Conjecture



HISTORIC CONTEXT

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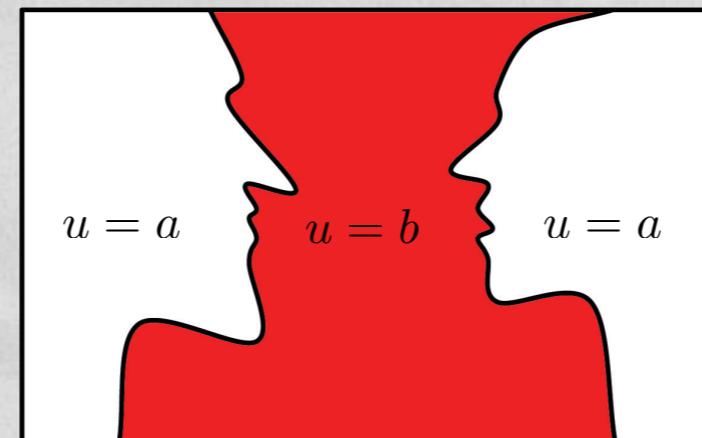
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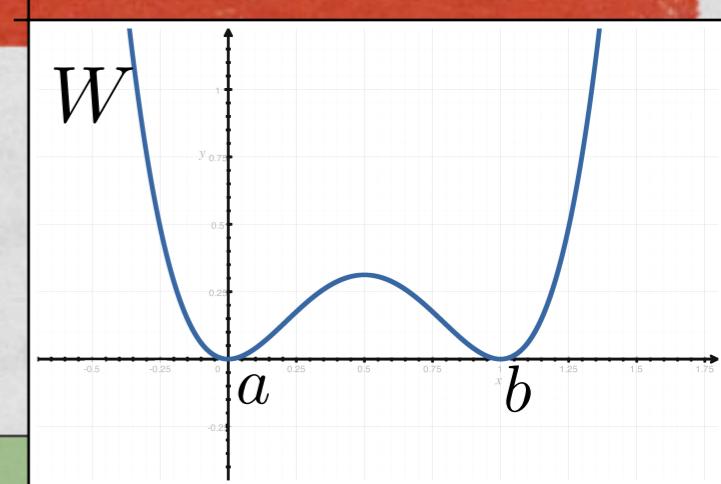
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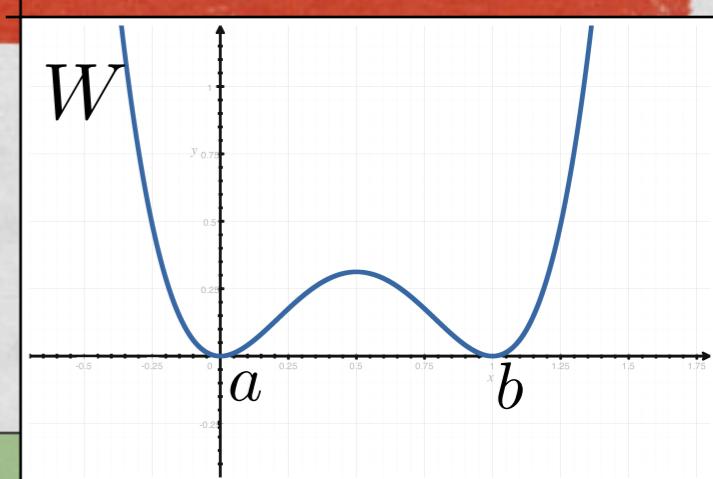
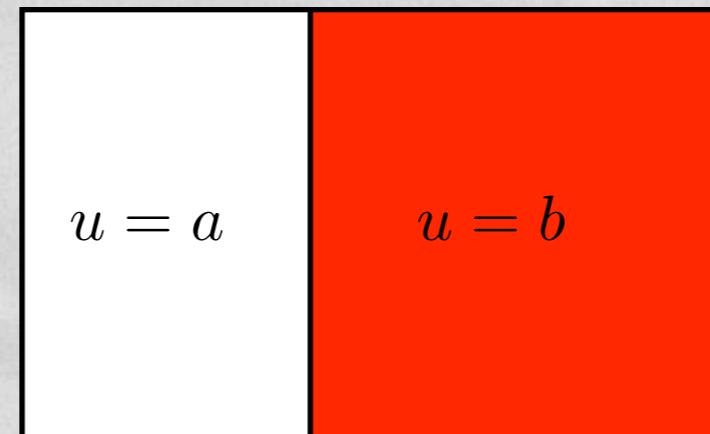
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HISTORIC CONTEXT

PHASE TRANSITIONS

Modica, Mortola '77

$$\left| \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \sin^2 \left(\frac{\pi u}{\varepsilon} \right) dx + \varepsilon \int_{\mathbb{R}^N} |Du|^2 dx \right|$$

HISTORIC CONTEXT

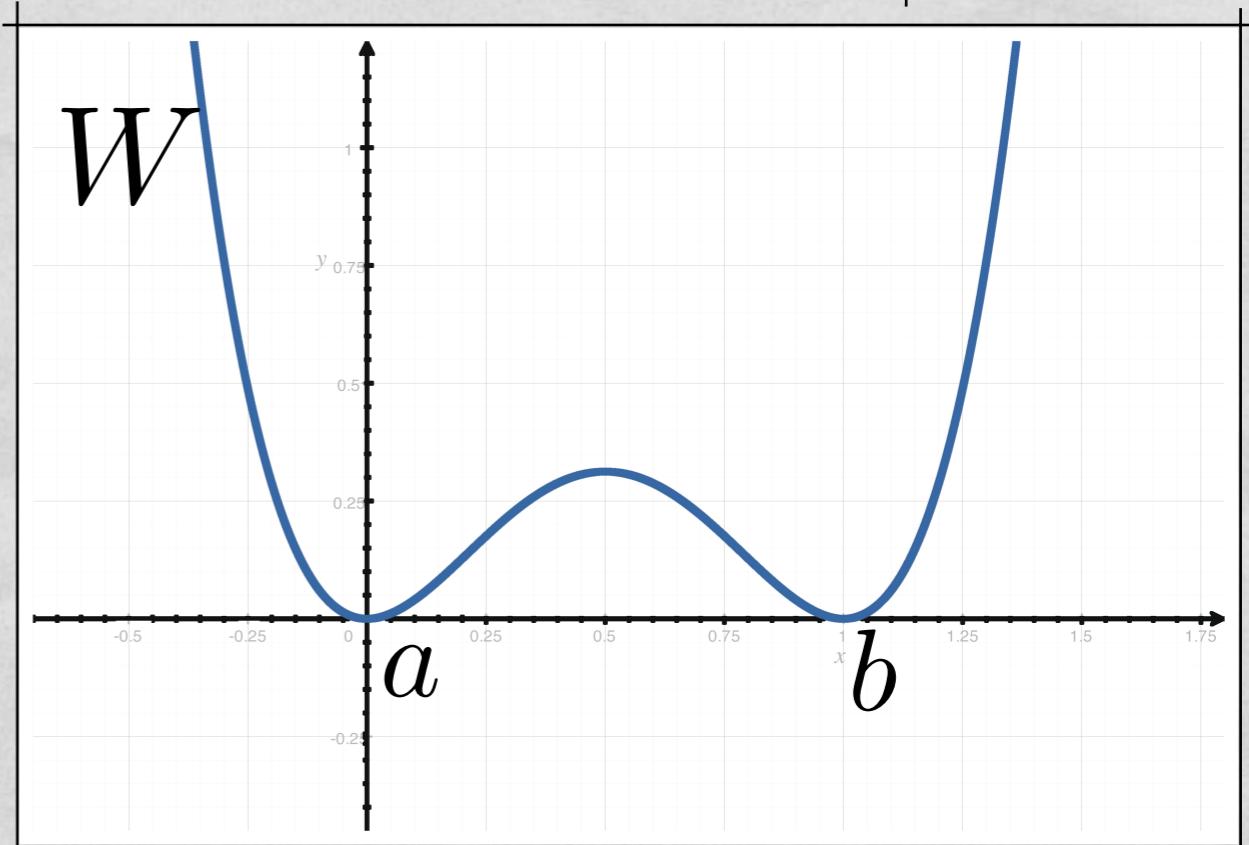
PHASE TRANSITIONS

Modica, Mortola '77

Modica '87, Sternberg '88

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^N} \sin^2 \left(\frac{\pi u}{\varepsilon} \right) dx + \varepsilon \int_{\mathbb{R}^N} |Du|^2 dx$$

$$\frac{1}{\varepsilon} \int_{\Omega} W(u) dx + \varepsilon \int_{\Omega} |Du|^2 dx$$



HISTORIC CONTEXT

PHASE TRANSITIONS

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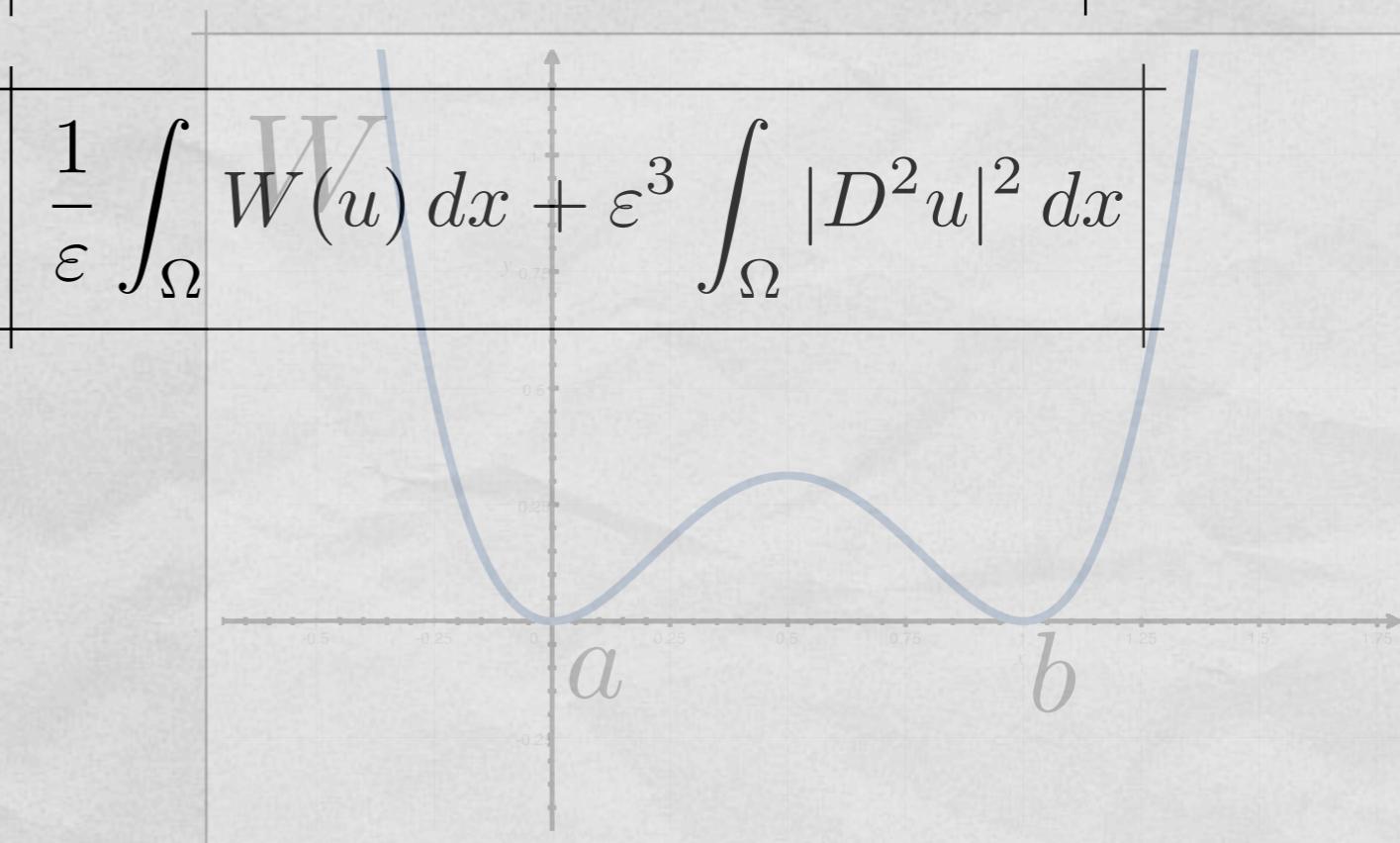
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Modica '87, Sternberg '88

$$\frac{1}{\varepsilon} \int_{\Omega} W(u) dx + \varepsilon \int_{\Omega} |Du|^2 dx$$

Fonseca, Mantegazza '00

$$\frac{1}{\varepsilon} \int_{\Omega} W(u) dx + \varepsilon^3 \int_{\Omega} |D^2u|^2 dx$$



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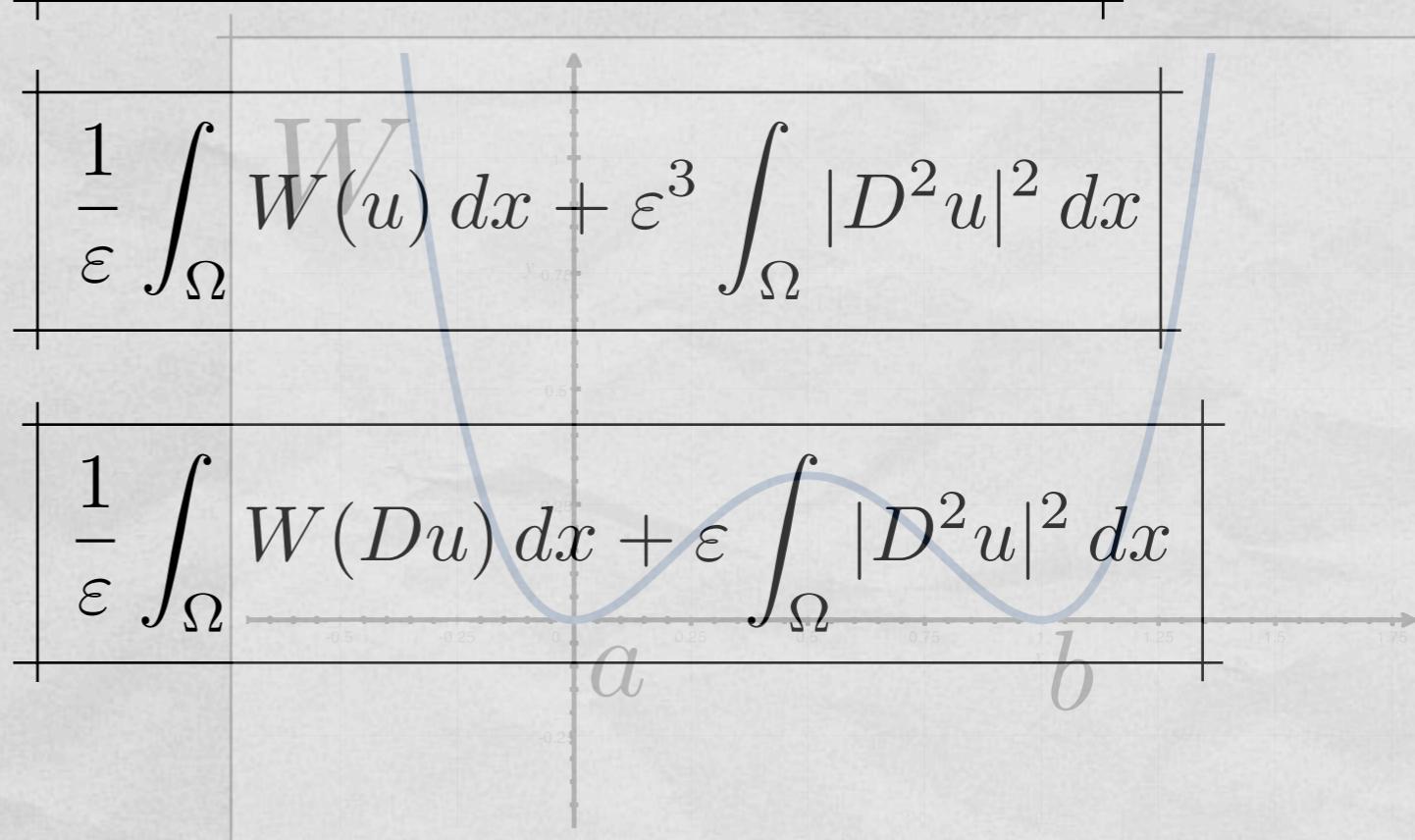
$$\frac{1}{\varepsilon} \int_{\Omega} W(u) dx + \varepsilon \int_{\Omega} |Du|^2 dx$$

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$$\frac{1}{\varepsilon} \int_{\Omega} W(u) dx + \varepsilon^3 \int_{\Omega} |D^2u|^2 dx$$

Conti, Fonseca, Leoni '02

$$\frac{1}{\varepsilon} \int_{\Omega} W(Du) dx + \varepsilon \int_{\Omega} |D^2u|^2 dx$$



HISTORIC CONTEXT

PHASE TRANSITIONS

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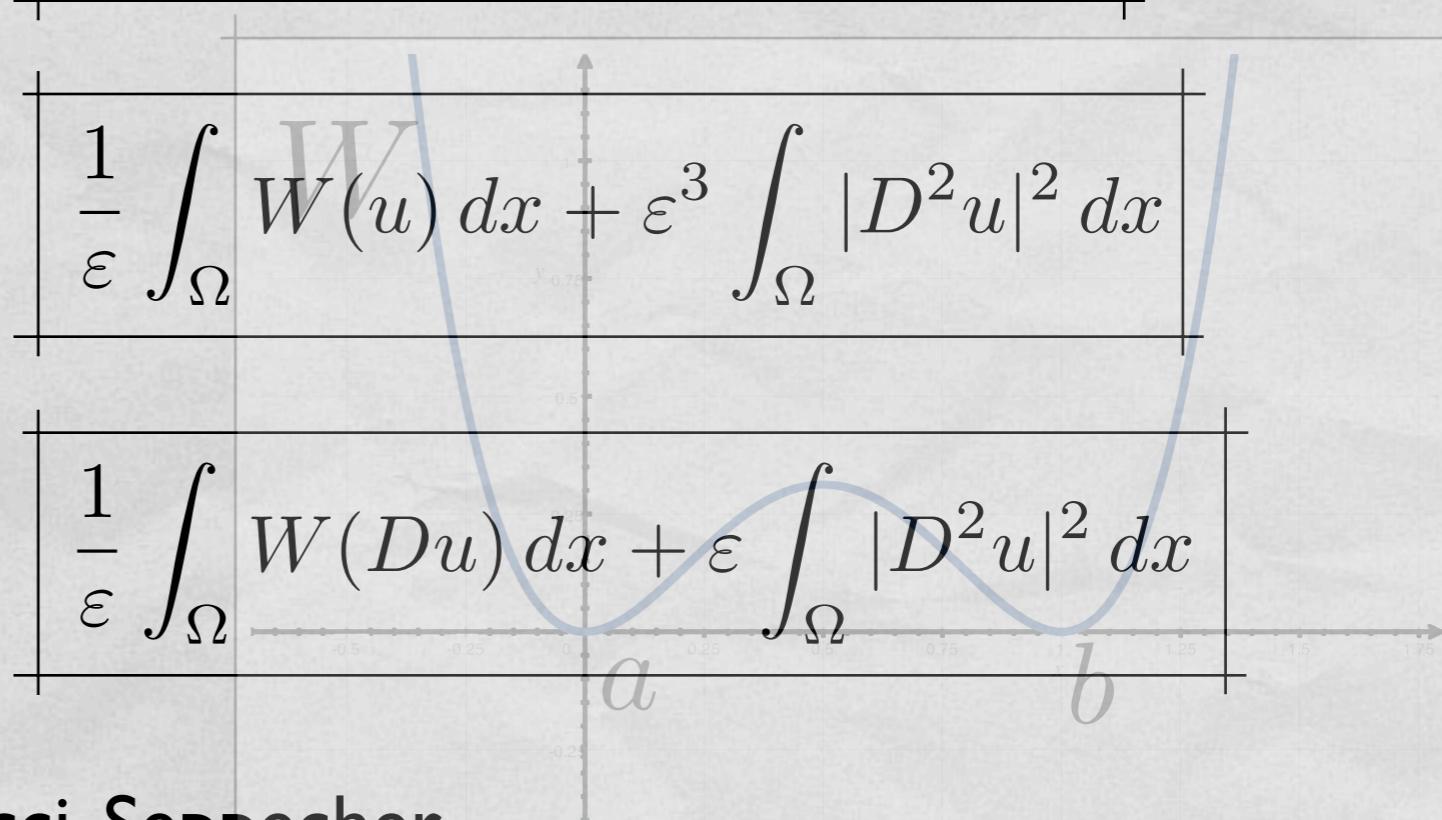
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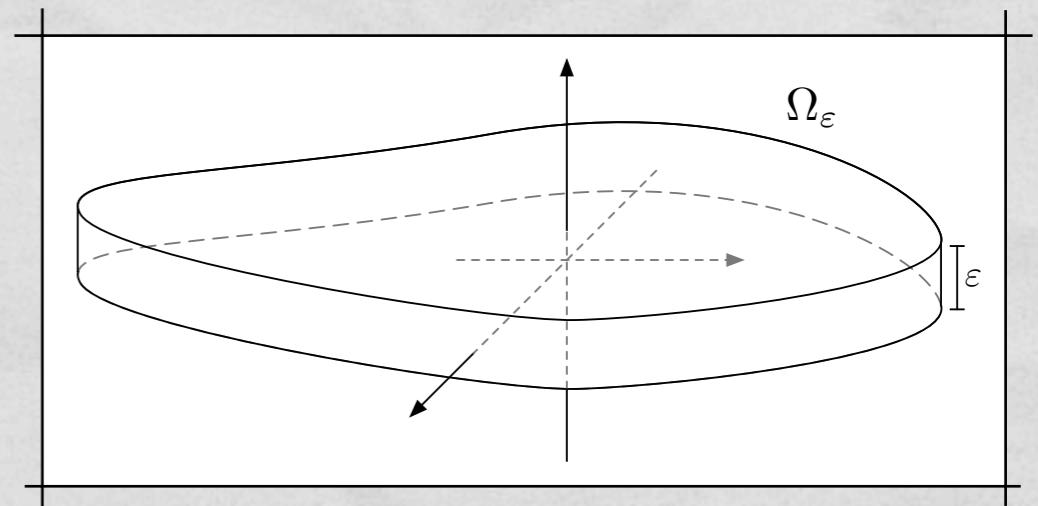
Alberti, Bouchitté, Garroni, Palatucci, Seppecher, ...



HISTORIC CONTEXT

THIN
FILMS

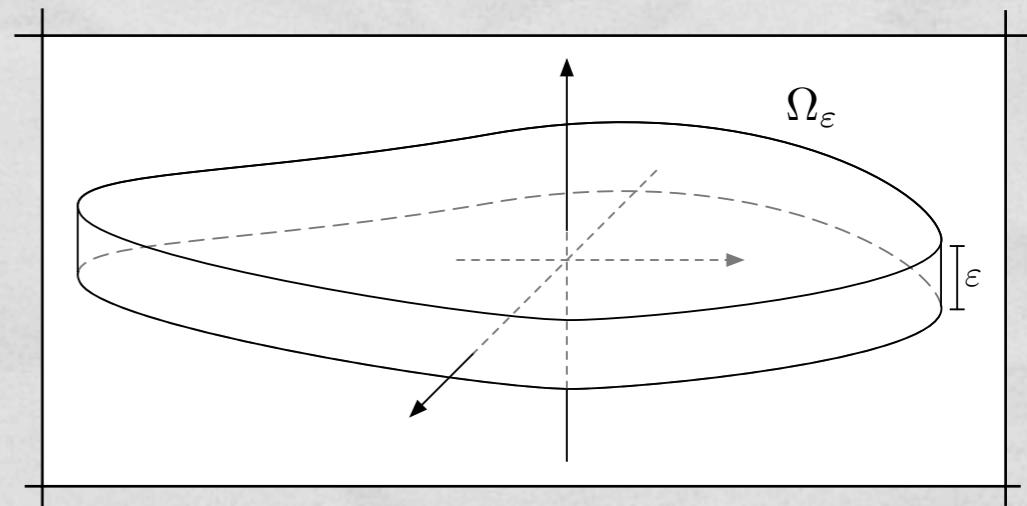
Instead of using the 3D Elasticity model on a very thin domain



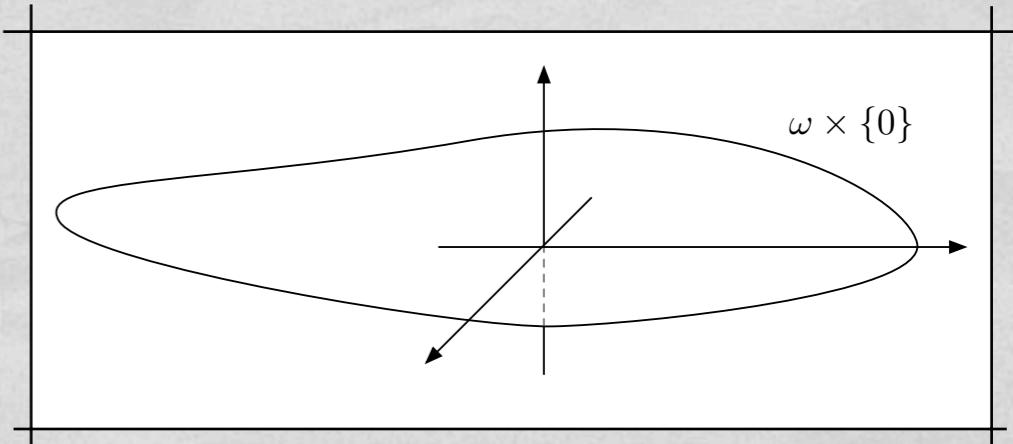
HISTORIC CONTEXT

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Instead of using the 3D Elasticity model on a very thin domain



Deduce a 2D model from the 3D Elasticity Equations by letting the height of the domain go to 0



HISTORIC CONTEXT

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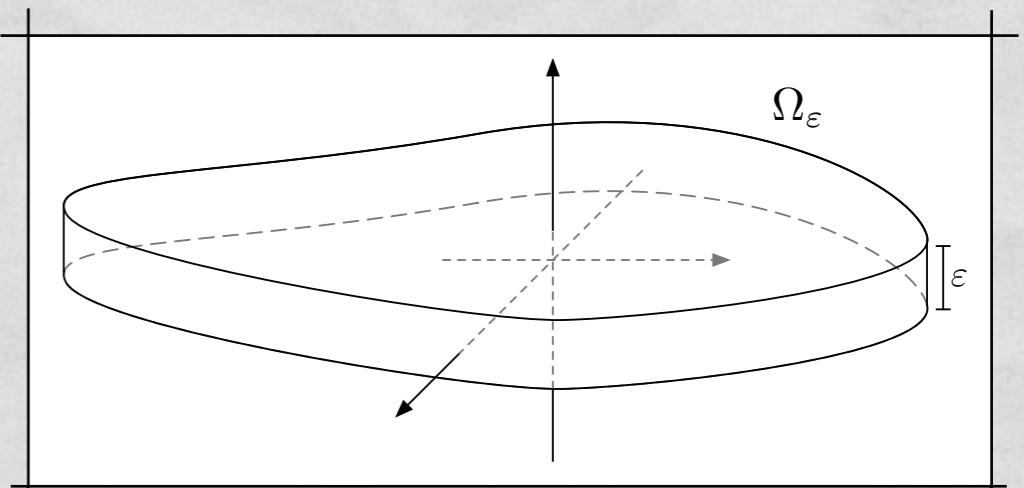
idea

3D Elasticity $\inf_{u \in H^1} \int_{\Omega_\varepsilon} W(\nabla u(\mathbf{x})) \, d\mathbf{x}$

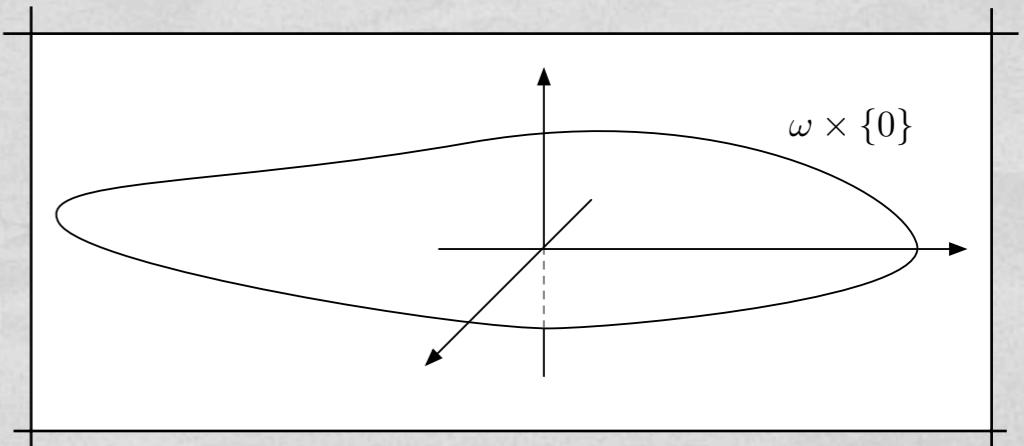
infimizing sequence $\{u_\varepsilon\}$

$$u_\varepsilon \xrightarrow{\varepsilon \rightarrow 0^+} u_0$$

Q: What is the limiting problem?
Q: How to characterize it?



$$\downarrow \quad \varepsilon \rightarrow 0^+$$



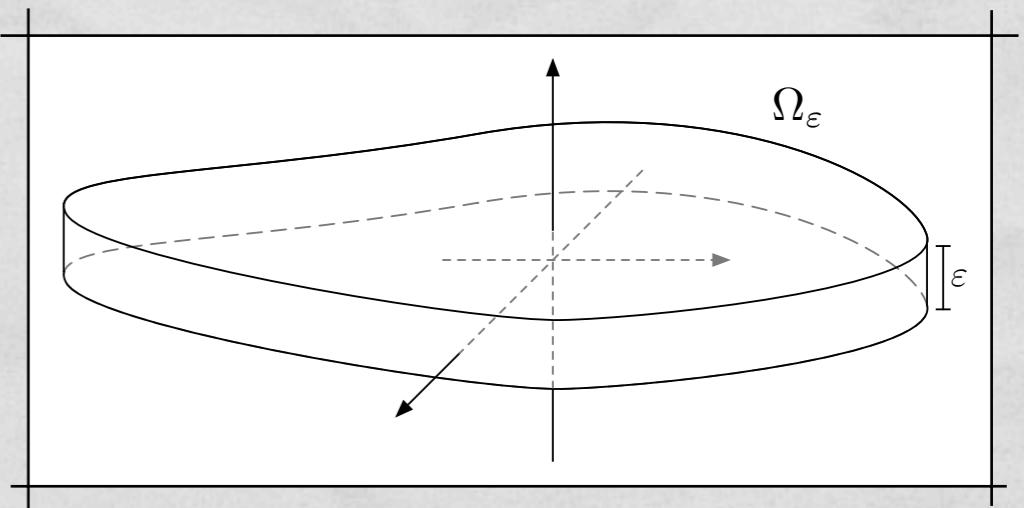
HISTORIC CONTEXT

THIN
FILMS

Tool: Γ - limit

Step I. Rescale

$$\inf_{u_{eps} \in H^1} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(\nabla u_\varepsilon(\mathbf{x})) \, d\mathbf{x}$$



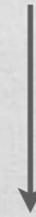
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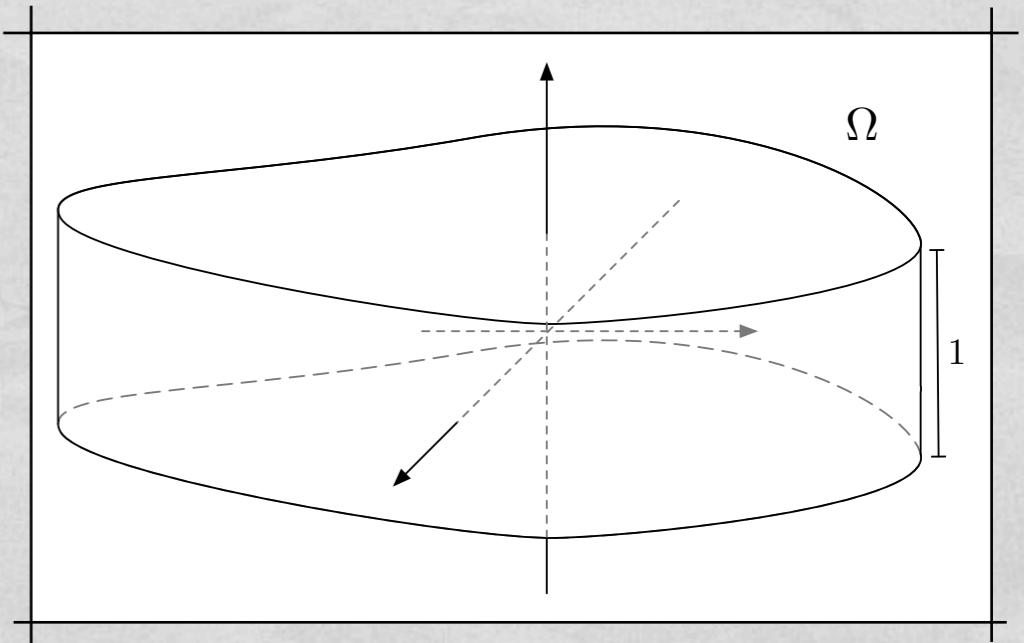
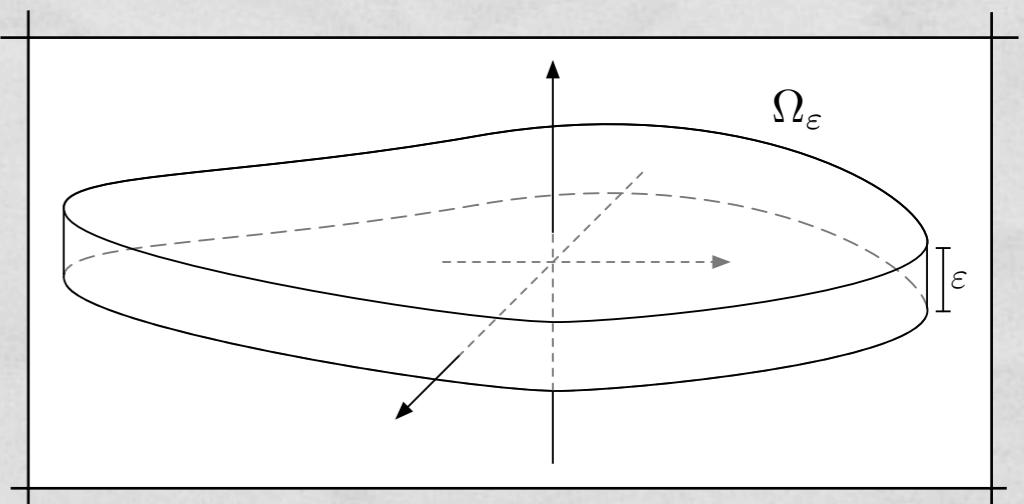
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$$\inf_{u_{eps} \in H^1} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(\nabla u_\varepsilon(\mathbf{x})) \, d\mathbf{x}$$



$$\inf_{u_\varepsilon \in H^1} \int_{\Omega} W(\nabla' u_\varepsilon(x) | \frac{1}{\varepsilon} \partial_3 u_\varepsilon(x)) \, dx$$



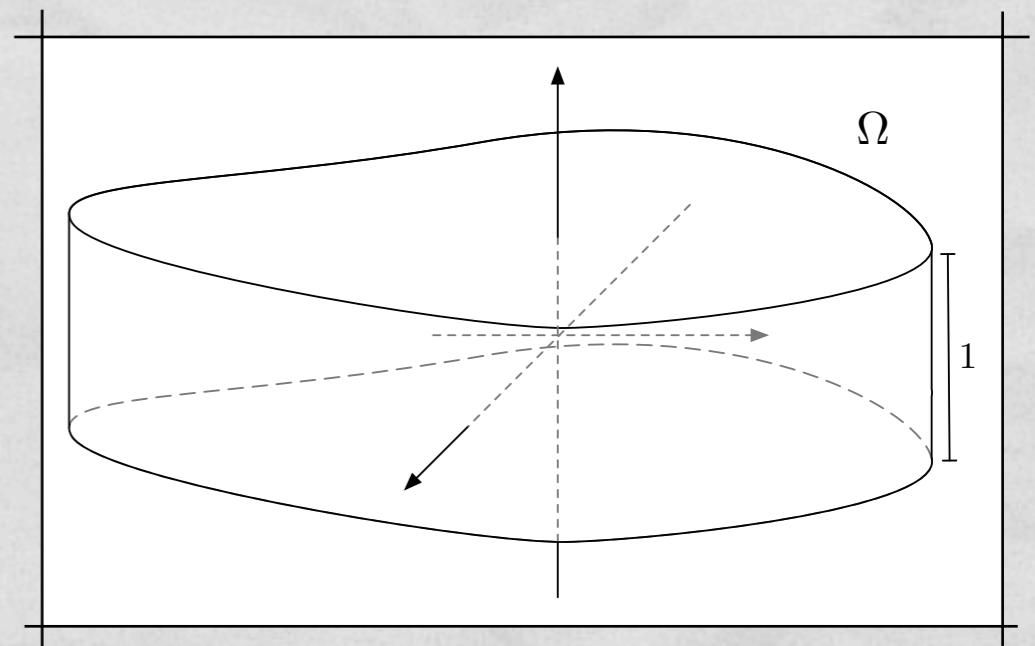
HISTORIC CONTEXT

THIN
FILMS

Tool: Γ - limit

Step 2. Γ - limit

$$\inf_{u_\varepsilon \in H^1} \int_{\Omega} W\left(\nabla' u_\varepsilon(x) \mid \frac{1}{\varepsilon} \partial_3 u_\varepsilon(x)\right) dx$$



HISTORIC CONTEXT

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FILMS

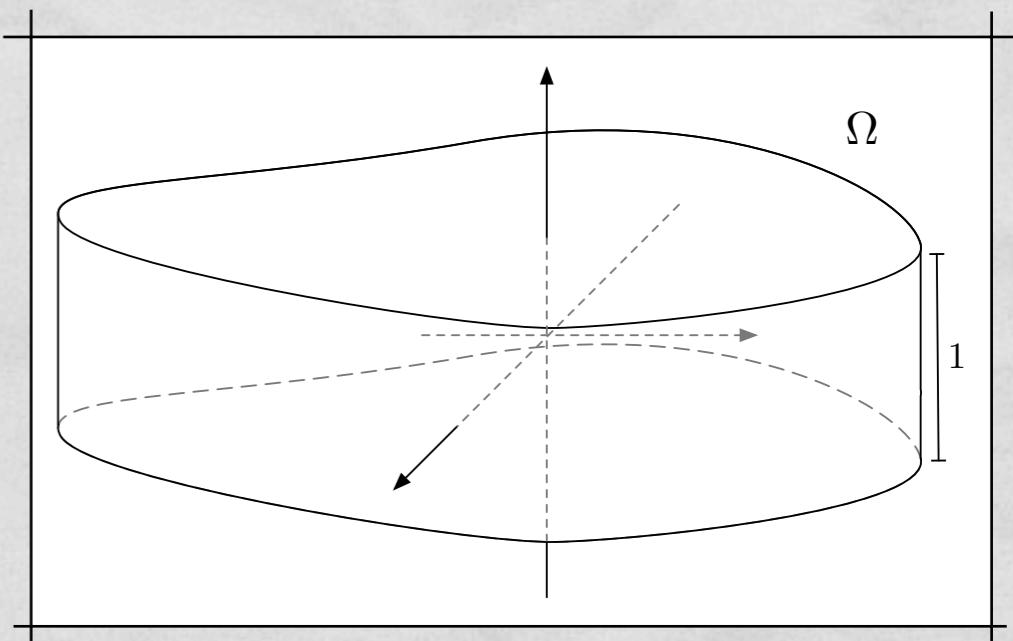
Tool: Γ - limit

Step 2. Γ - limit

$$\inf_{u_\varepsilon \in H^1} \int_{\Omega} W\left(\nabla' u_\varepsilon(x) \mid \frac{1}{\varepsilon} \partial_3 u_\varepsilon(x)\right) dx$$

↓
 Γ - limit

$$\min_{\substack{u \in H^1 \\ \partial_3 u = 0}} \int_{\Omega} QW_0(\nabla' u(x)) dx$$



$$W_0(\xi') := \min_{z \in \mathbb{R}^3} W(\xi' | z)$$

HISTORIC CONTEXT

THIN
FILMS

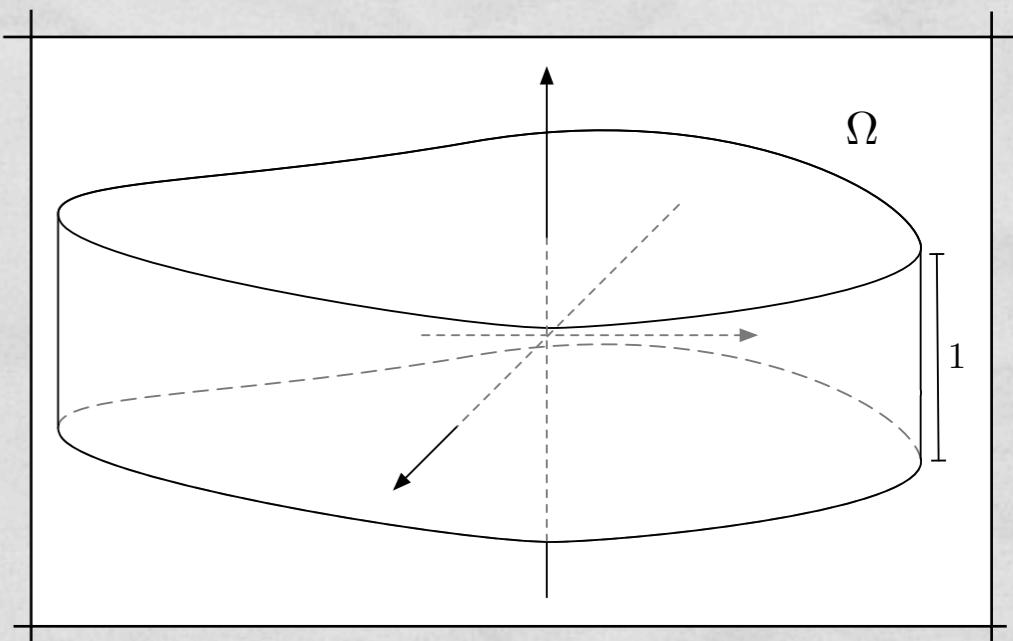
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HISTORIC CONTEXT

THIN
FILMS

Ciarlet, Destuynder '78

Kohn, Vogelius '85

Acerbi, Buttazzo '86

Acerbi, Buttazzo, Percivale '88

Le Dret, Raoult '93

Bhattacharya, James '99

Braides, Bouchitté, Fonseca, Francfort, Friesecke, Leoni, Müller, ...

PHASE TRANSITIONS FOR THIN FILMS

Γ -LIMIT

$I_\varepsilon : X \longrightarrow [0, \infty]$, X metric space

$I_\varepsilon(u)$ Γ -converges to $I_0(u)$ in the topology of X

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(Γ- lim inf) $\forall u_\varepsilon \xrightarrow{X} u, I_0(u) \leq \liminf I_\varepsilon(u_\varepsilon)$

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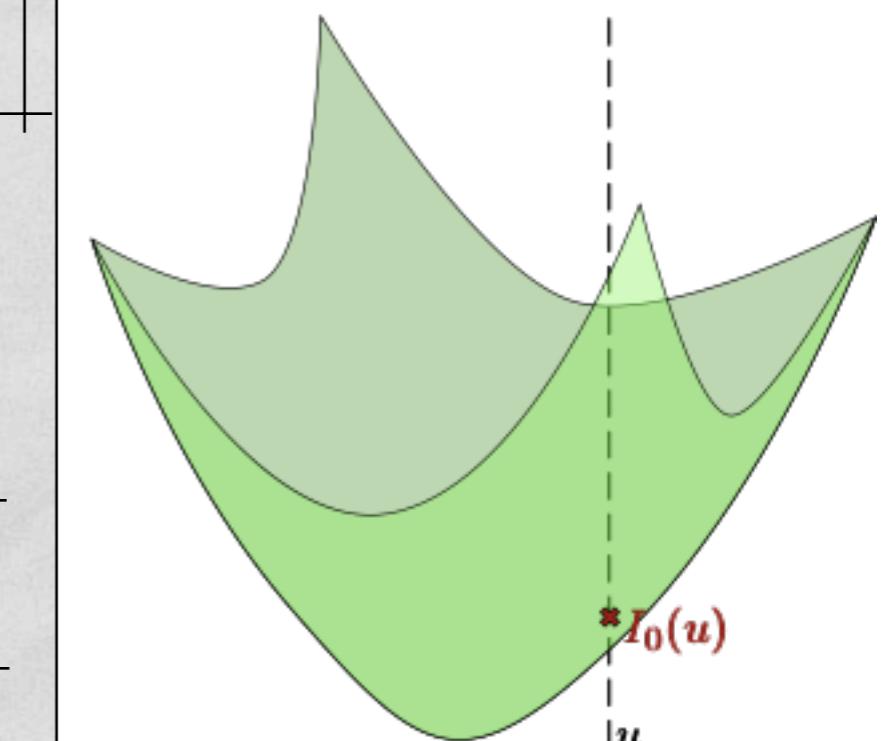
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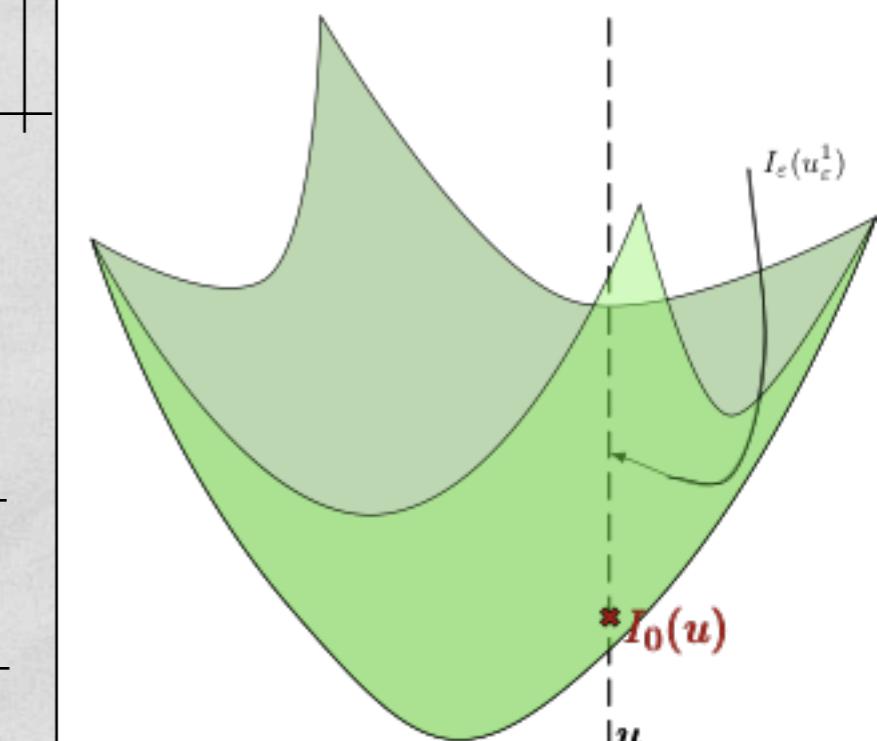
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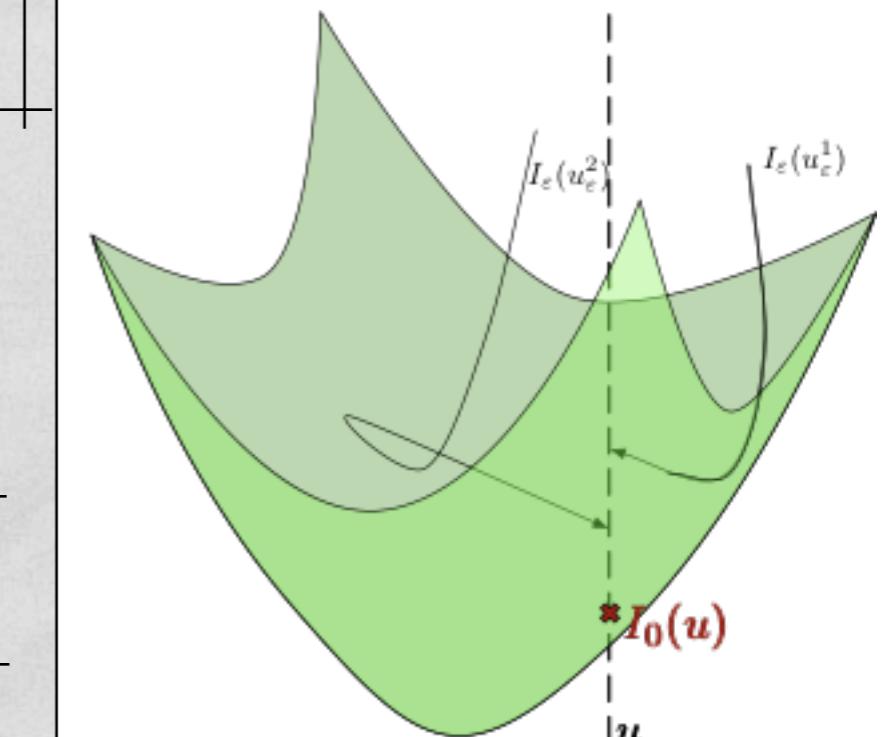
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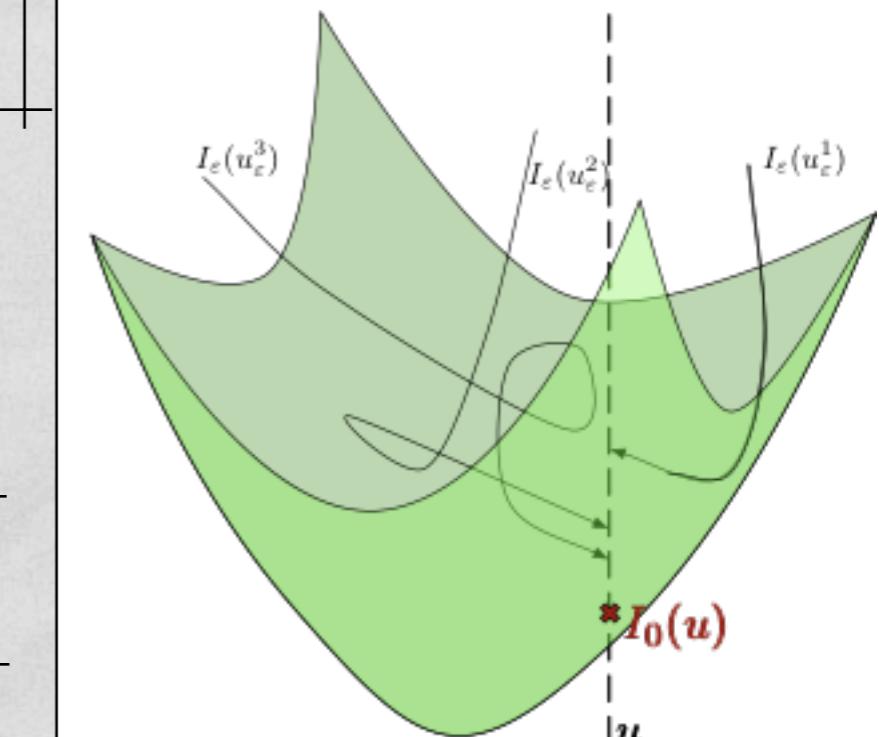
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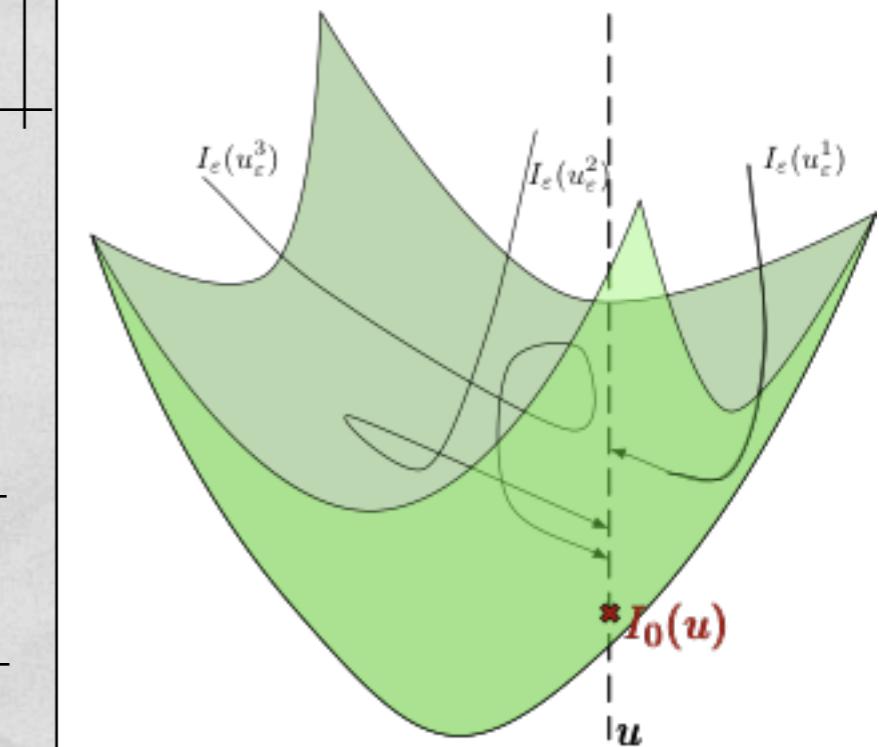
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(Γ - lim inf) $\forall u_\varepsilon \xrightarrow{X} u, I_0(u) \leq \liminf I_\varepsilon(u_\varepsilon)$

(Γ - lim sup) $\exists v_\varepsilon \xrightarrow{X} u, I_\varepsilon(v_\varepsilon) \rightarrow I_0(u)$



PHASE TRANSITIONS FOR THIN FILMS

Γ -LIMIT

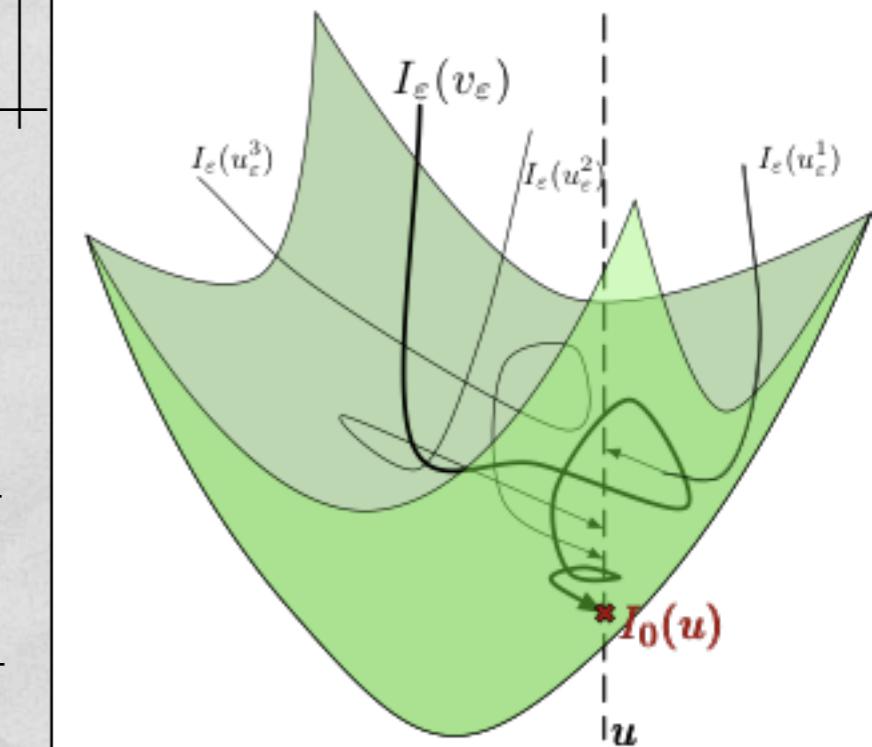
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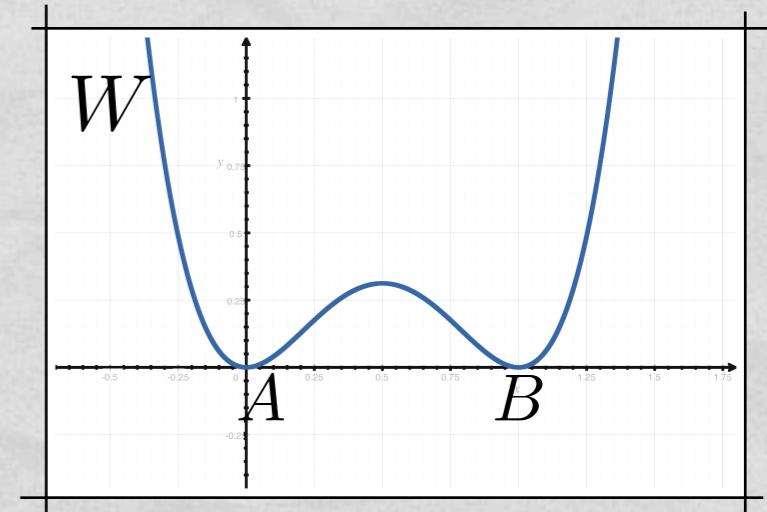
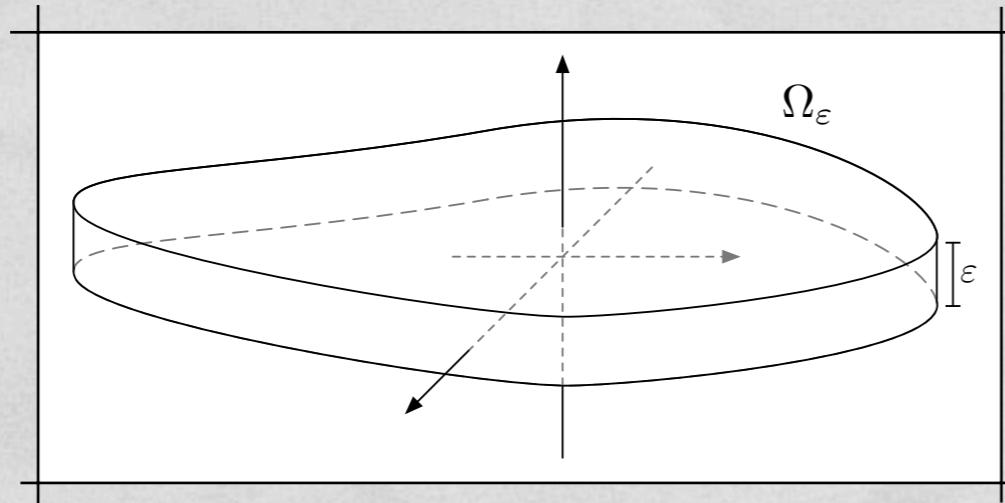
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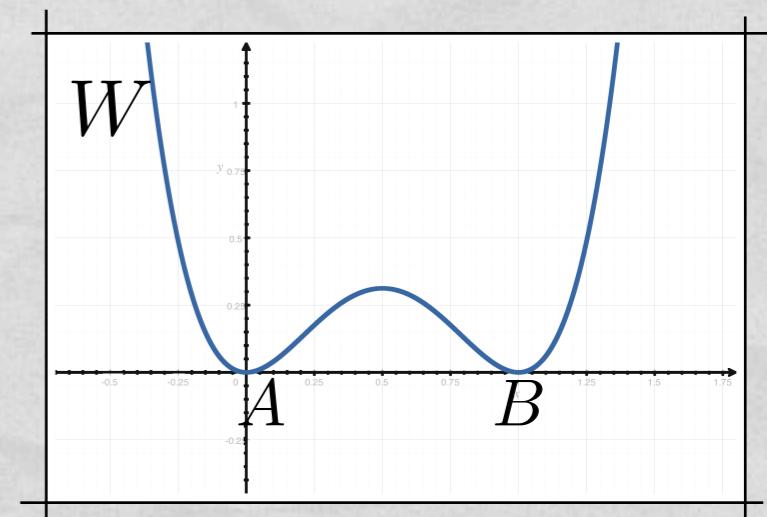
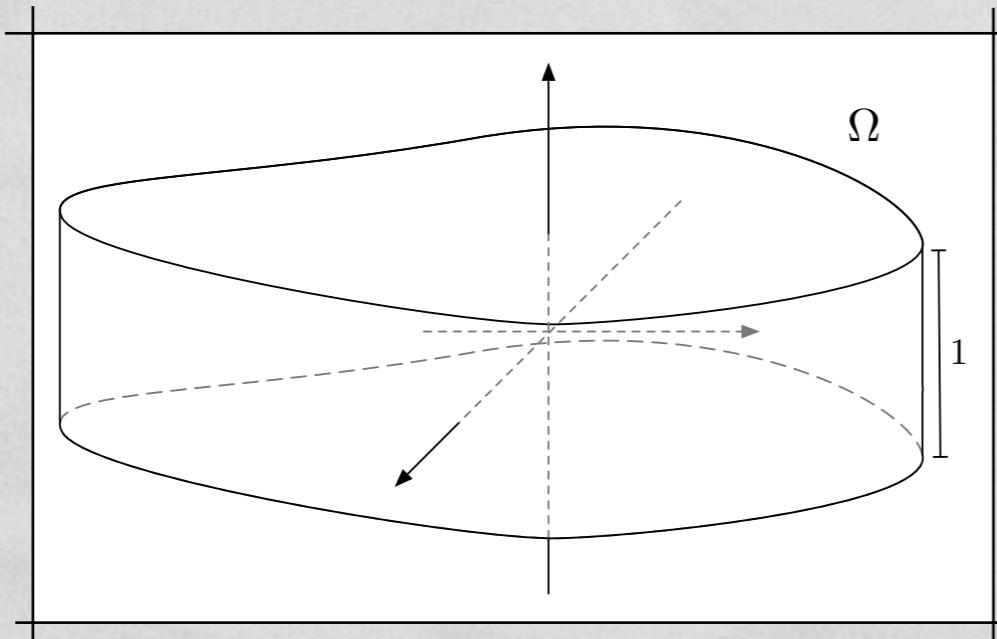
PHASE TRANSITIONS FOR THIN FILMS

$$\boxed{\mathbf{I}_\varepsilon^\gamma(\mathbf{u}) := \frac{1}{\varepsilon} \left(\frac{1}{\varepsilon^\gamma} \int_{\Omega_\varepsilon} W(\nabla \mathbf{u}) \, d\mathbf{x} + \varepsilon^\gamma \int_{\Omega_\varepsilon} |\nabla^2 \mathbf{u}|^2 \, d\mathbf{x} \right)}$$



PHASE TRANSITIONS FOR THIN FILMS

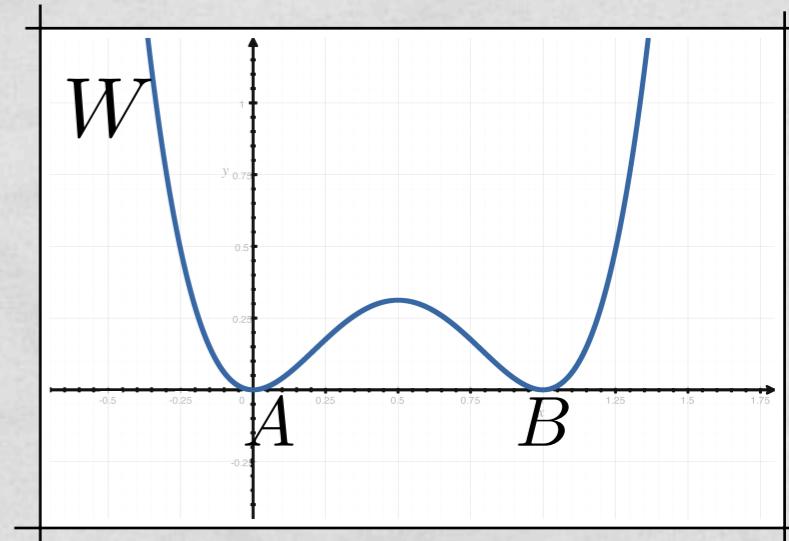
$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

$W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ continuous double-well potential

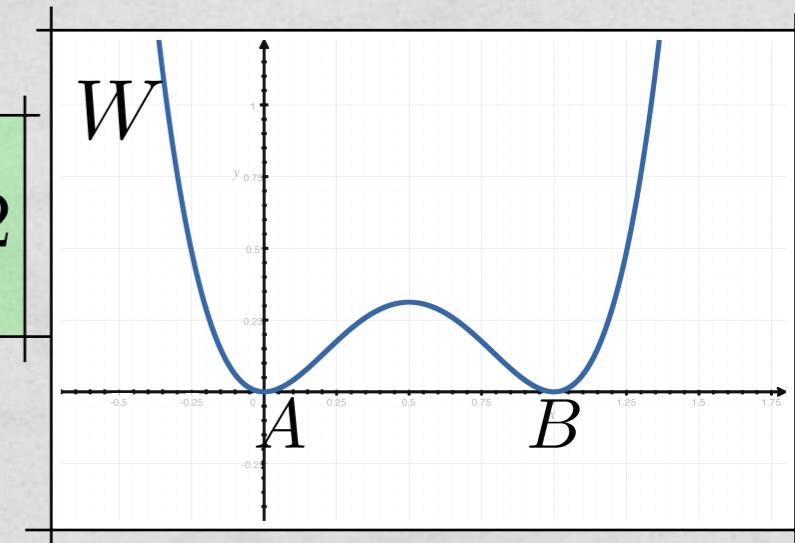


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Hypotheses on W

$W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ continuous double-well potential

$$(H_1) \quad \frac{1}{C_1} |\xi|^p - C_1 \leq W(\xi) \leq C_1 |\xi|^p + C_1, p \geq 2$$



PHASE TRANSITIONS IN THIN FILMS

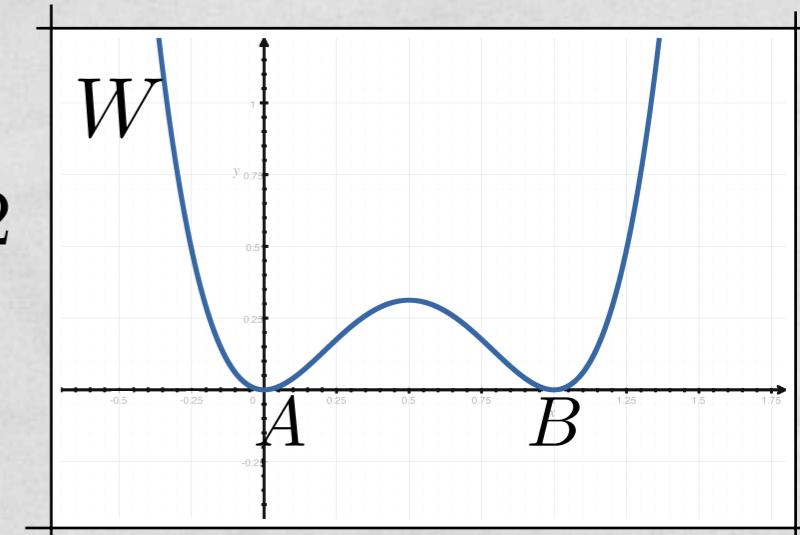
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$$A' - B' = a \otimes \nu \text{ rank-one connected}$$



PHASE TRANSITIONS IN THIN FILMS

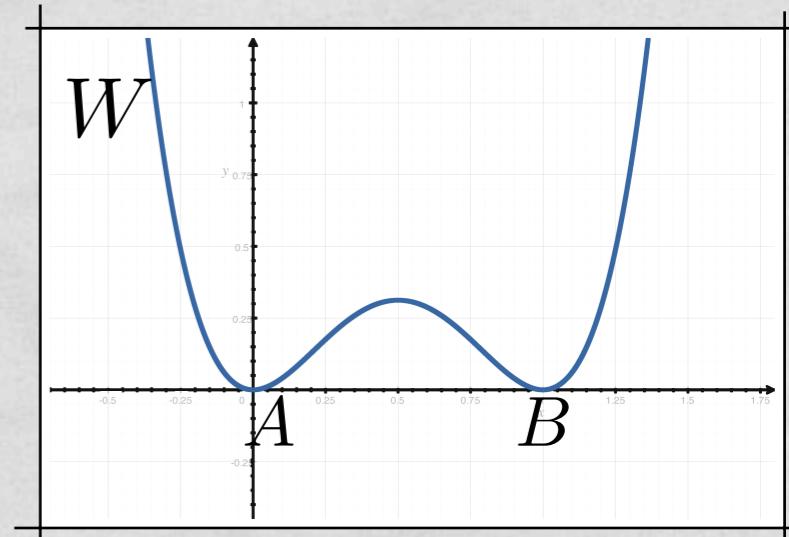
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$$(H_3) \quad \frac{1}{C_2} \operatorname{dist}(\xi, \{A, B\})^p \leq W(\xi) \leq C_2 \operatorname{dist}(\xi, \{A, B\})^p$$

if $\operatorname{dist}(\xi, \{A, B\}) \leq \rho$

PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

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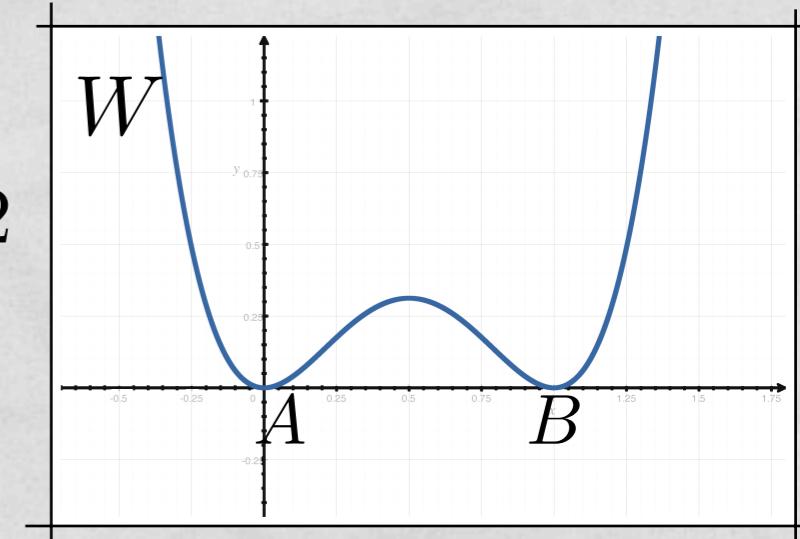
$A' - B' = a \otimes \nu$ rank-one connected

$$(H_3) \quad \frac{1}{C_2} \operatorname{dist}(\xi, \{A, B\})^p \leq W(\xi) \leq C_2 \operatorname{dist}(\xi, \{A, B\})^p$$

if $\operatorname{dist}(\xi, \{A, B\}) \leq \rho$

$$(H_4) \quad \begin{array}{l} \text{best path from } A' \text{ to } B' \text{ does not depend on } \nu^\perp \\ \text{if } A' \neq B' \end{array}$$

$W = W(|\xi'|, \xi_3)$ if $A' = B', A_3 \neq B_3$

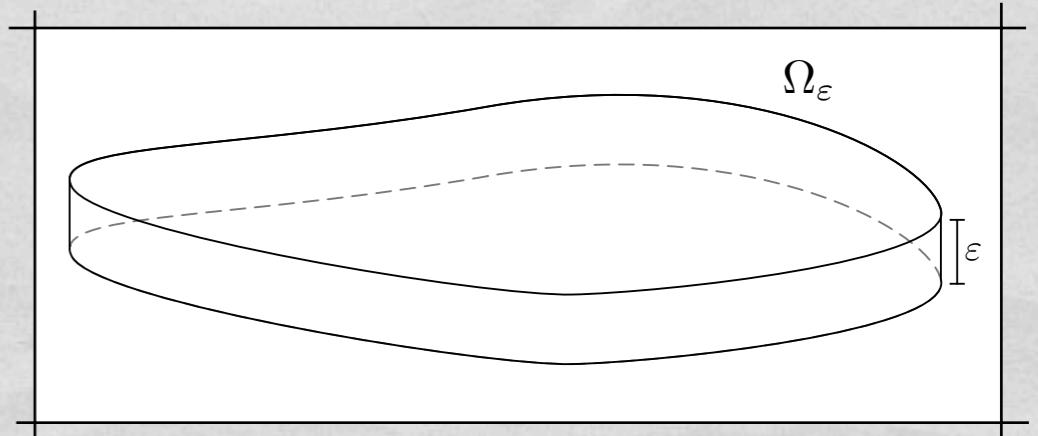


PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

ε height of the domain



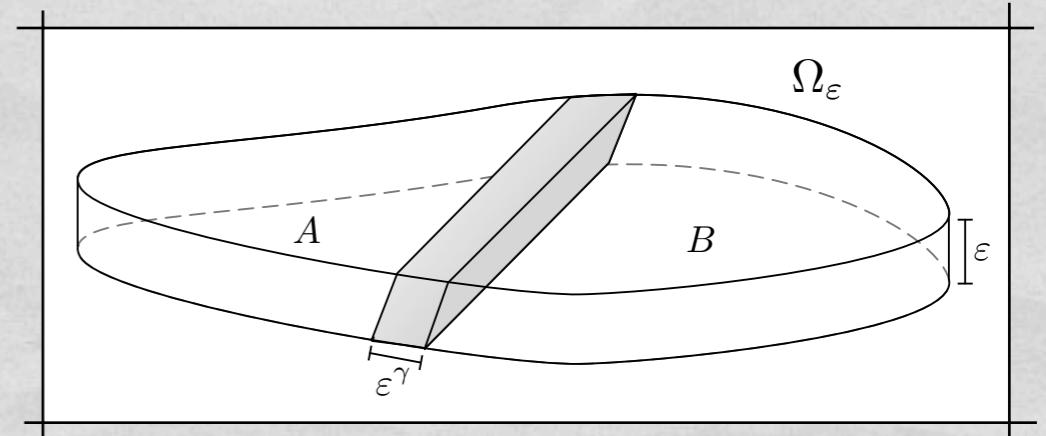
PHASE TRANSITIONS FOR THIN FILMS

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ε height of the domain

ε^γ width of the transition layer



PHASE TRANSITIONS FOR THIN FILMS

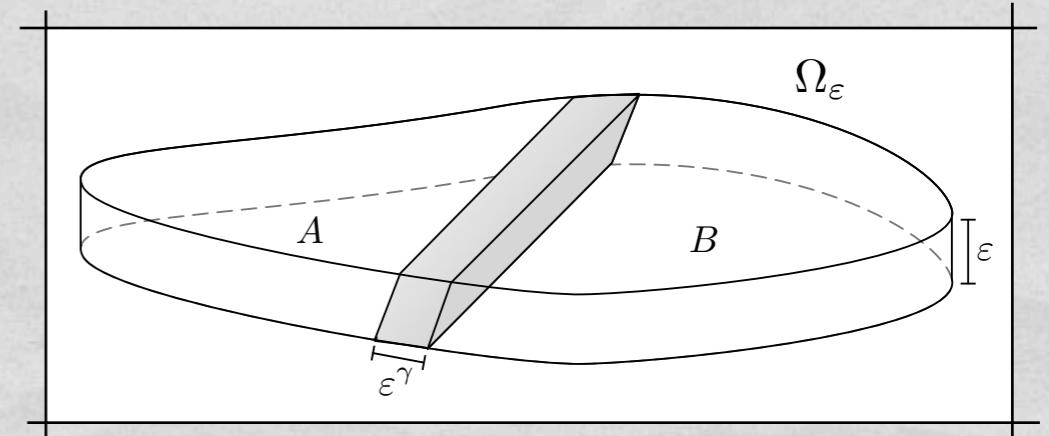
Identify Regimes

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

Three Regimes:

$$\begin{array}{|c|} \hline \gamma = 1 \\ \hline \end{array}$$

Critical Case



PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

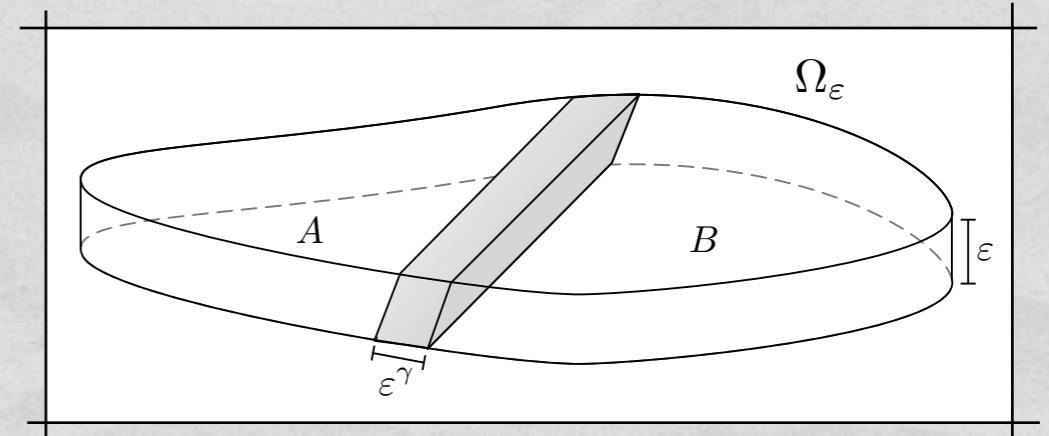
Three Regimes:

$$\boxed{\gamma = 1}$$

Critical Case

$$\boxed{\gamma < 1}$$

Subcritical Case



PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

Three Regimes:

$$\boxed{\gamma = 1}$$

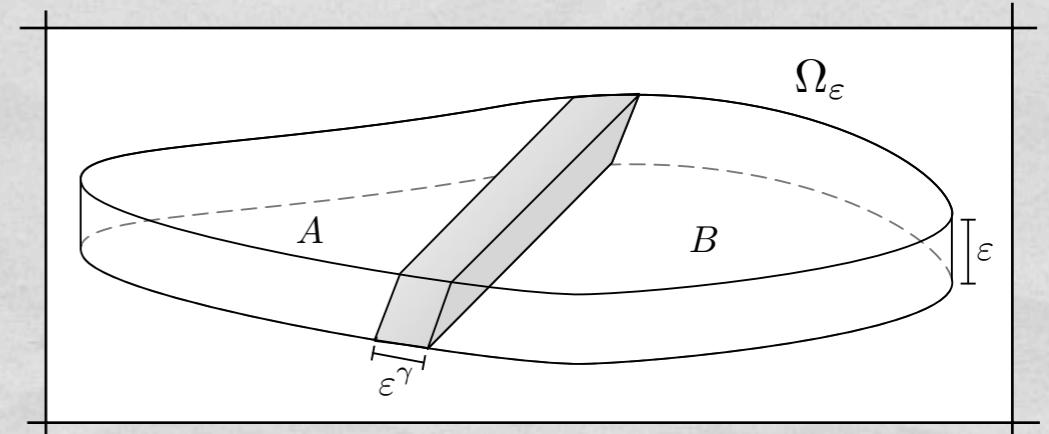
Critical Case

$$\boxed{\gamma < 1}$$

Subcritical Case

$$\boxed{\gamma > 1}$$

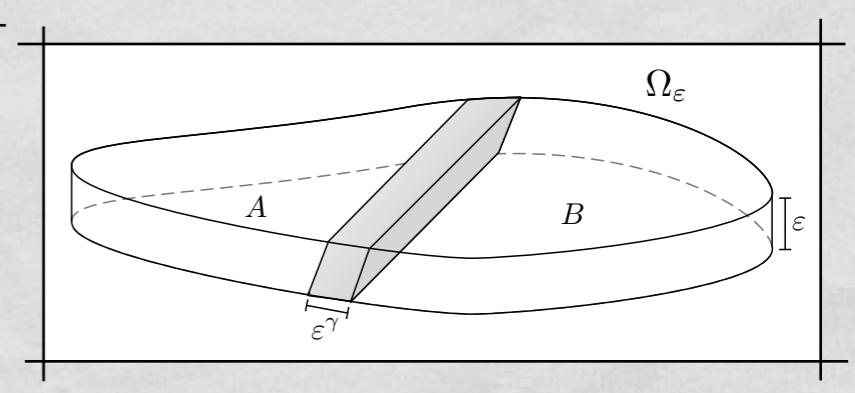
Supercritical Case



PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

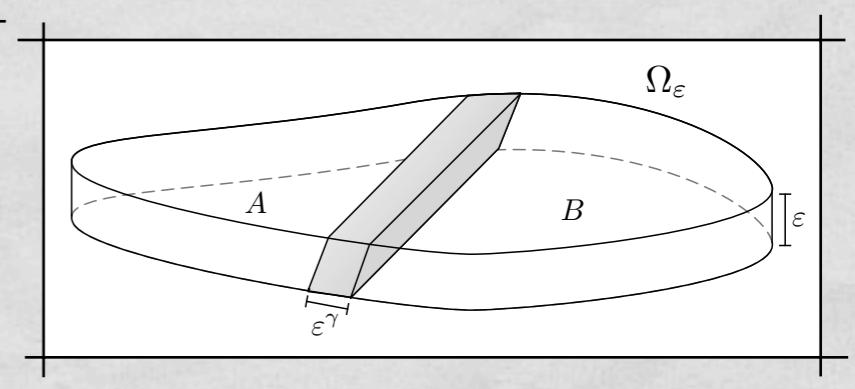


Results:

PHASE TRANSITIONS FOR THIN FILMS

Results

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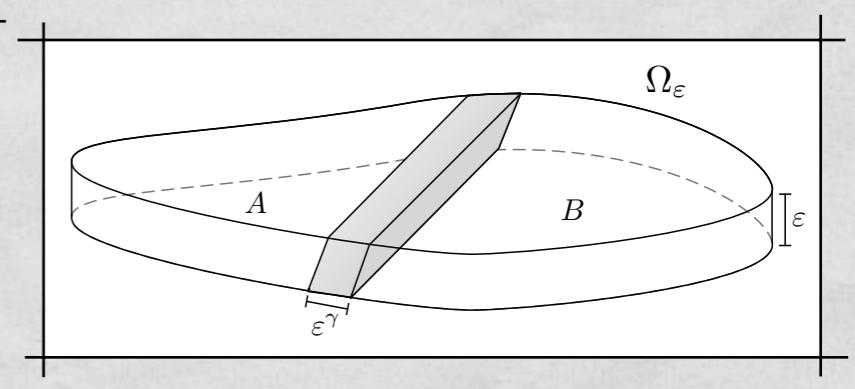
Results:

$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

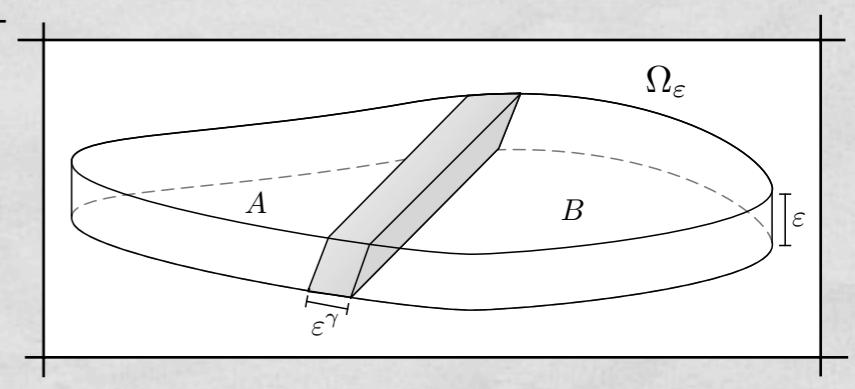
$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

$$A' \neq B'$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

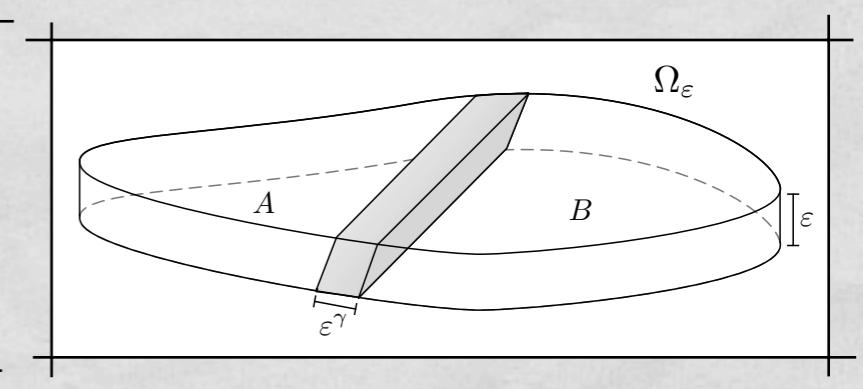
$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

$$\begin{array}{c} \xrightarrow{\hspace{1cm}} A' \neq B' \\ \xrightarrow{\hspace{1cm}} A' = B', A_3 \neq B_3 \end{array}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

$$A' \neq B'$$

$$A' = B', A_3 \neq B_3$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\begin{array}{|c|c|}\hline \nabla' u & \frac{1}{\varepsilon} \partial_3 u \\ \hline \end{array}\right) dx + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

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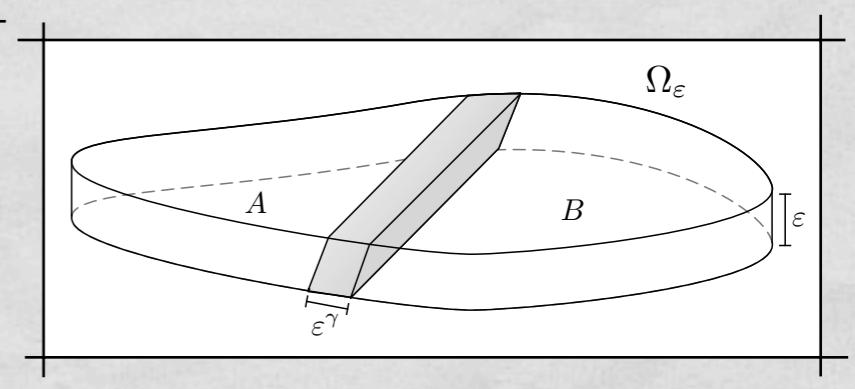
$$A = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right)$$

A' A_3

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

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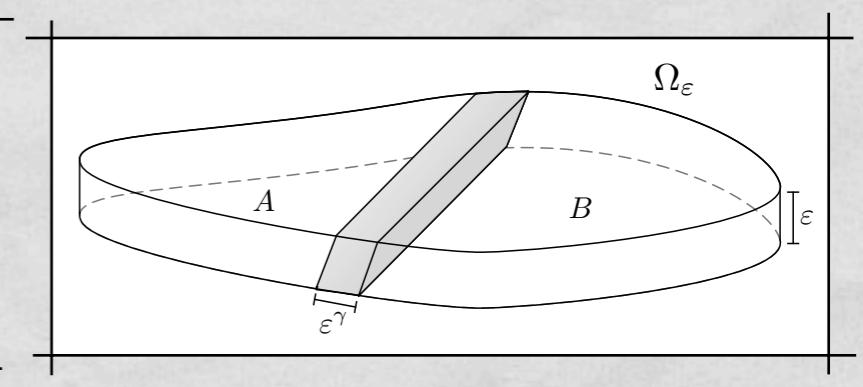
$$A' = B', A_3 \neq B_3$$

$$\boxed{\gamma < 1} \longrightarrow \Gamma\text{-limit}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

$$\begin{array}{c} A' \neq B' \\ \swarrow \quad \searrow \\ A' = B', A_3 \neq B_3 \end{array}$$

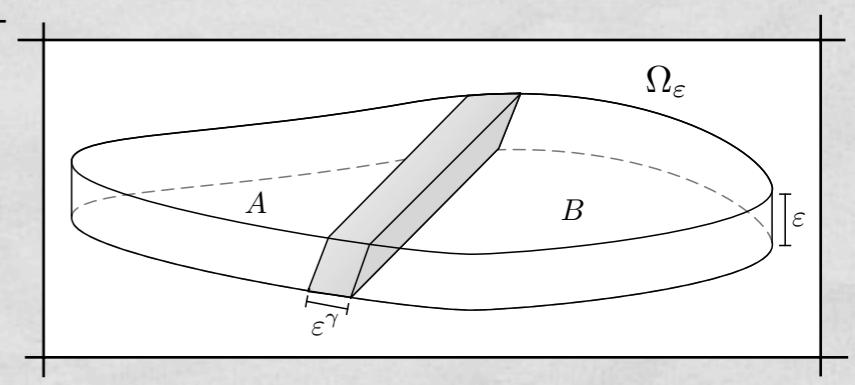
$$\boxed{\gamma < 1} \longrightarrow \Gamma\text{-limit}$$

$$\boxed{\gamma > 1}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

$$\boxed{\gamma = 1} \longrightarrow \text{Γ-limit}$$

$$A' \neq B'$$

$$A' = B', A_3 \neq B_3$$

$$\boxed{\gamma < 1} \longrightarrow \text{Γ-limit}$$

$$A - B = a \otimes \nu \quad \text{rank-one connected}$$

$$A' \neq B'$$

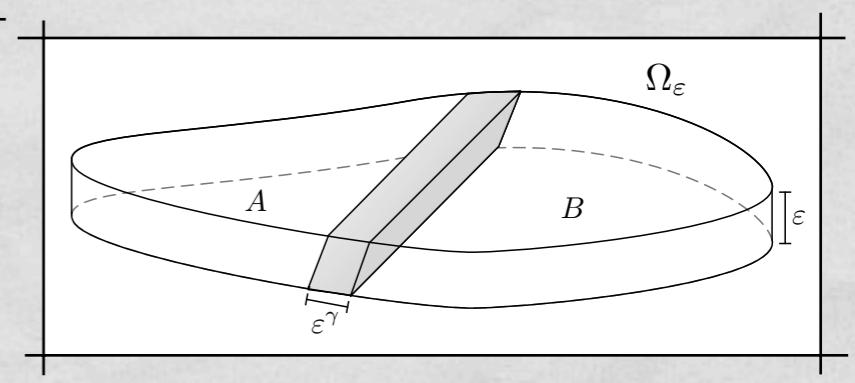
$$\boxed{\gamma > 1}$$

$$\longrightarrow \text{Γ-limit}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



Results:

$$\boxed{\gamma = 1} \longrightarrow \text{Γ-limit}$$

$$A' \neq B'$$

$$A' = B', A_3 \neq B_3$$

$$\boxed{\gamma < 1} \longrightarrow \text{Γ-limit}$$

$$A - B = a \otimes \nu \quad \text{rank-one connected}$$

$$\longrightarrow \text{Γ-limit}$$

$$\boxed{\gamma > 1}$$

$$A' \neq B'$$

$$\gamma > p, A' = B' \longrightarrow \text{trivial } \Gamma\text{-limit}$$

PHASE TRANSITIONS FOR THIN FILMS

Rigidity

Ball, James '87

$$\boxed{u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \nabla u \in \{A, B\}}$$

$$\Rightarrow \boxed{A - B = a \otimes \nu \quad \text{rank-one connected}}$$

PHASE TRANSITIONS FOR THIN FILMS

Rigidity

Ball, James '87

$$\boxed{\begin{array}{l} u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \nabla u \in \{A, B\} \end{array}}$$

$$\Rightarrow \boxed{\begin{array}{l} A - B = a \otimes \nu \quad \text{rank-one connected} \\ u(x) = \gamma_0 + Ax + h(x)a \\ \nabla h(x) = \chi_{\{\nabla u = B\}}(x)\nu \end{array}}$$

PHASE TRANSITIONS FOR THIN FILMS

Rigidity

Ball, James '87

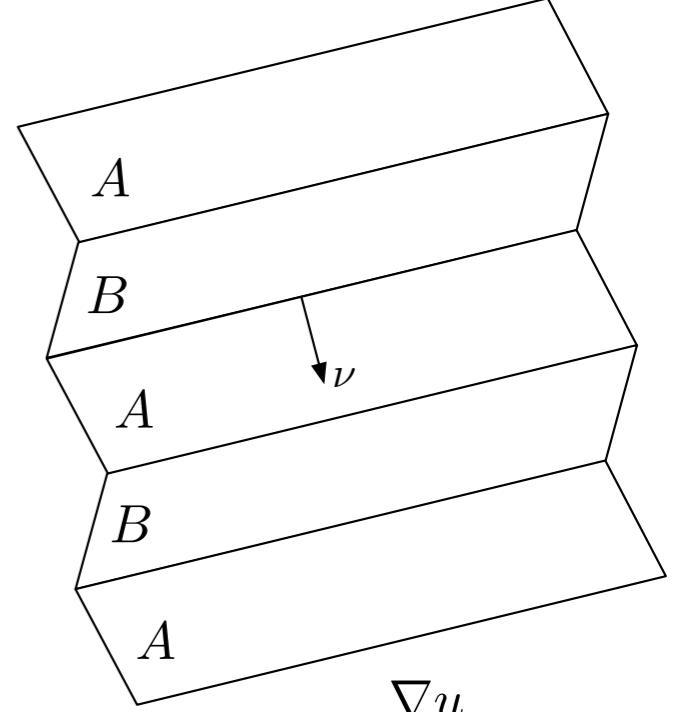
$$u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$$

$$\nabla u \in \{A, B\}$$

$$A - B = a \otimes \nu \quad \text{rank-one connected}$$

$$\Rightarrow u(x) = \gamma_0 + Ax + h(x)a$$

$$\nabla h(x) = \chi_{\{\nabla u = B\}}(x)\nu$$



PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$

Critical Case

Rigidity

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \begin{array}{l} u_\varepsilon \rightarrow u \\ \frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b \end{array} \quad (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$

Critical Case

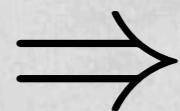
Rigidity

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

$$u_\varepsilon \rightarrow u$$

$$\frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b$$

$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$



$$u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$$

$$\nabla u \in \{(A'|0), (B'|0)\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Rigidity

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

$u_\varepsilon \rightarrow u$
 $\frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b$

$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

$$\Rightarrow \boxed{u \in W^{1,\infty}(\Omega; \mathbb{R}^3)} \\ \boxed{\nabla u \in \{(A'|0), (B'|0)\}}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

\$A'\$ \$A_3\$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$

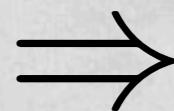
Critical Case

Rigidity

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

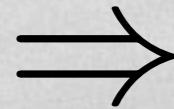
$$\begin{aligned} u_\varepsilon &\rightarrow u \\ \frac{1}{\varepsilon} \partial_3 u_\varepsilon &\rightarrow b \end{aligned}$$

$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$



$$\begin{aligned} u &\in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \nabla u &\in \{(A'|0), (B'|0)\} \end{aligned}$$

Ball, James '87



$$A' - B' = a \otimes \nu \quad \text{rank-one connected}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Rigidity

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

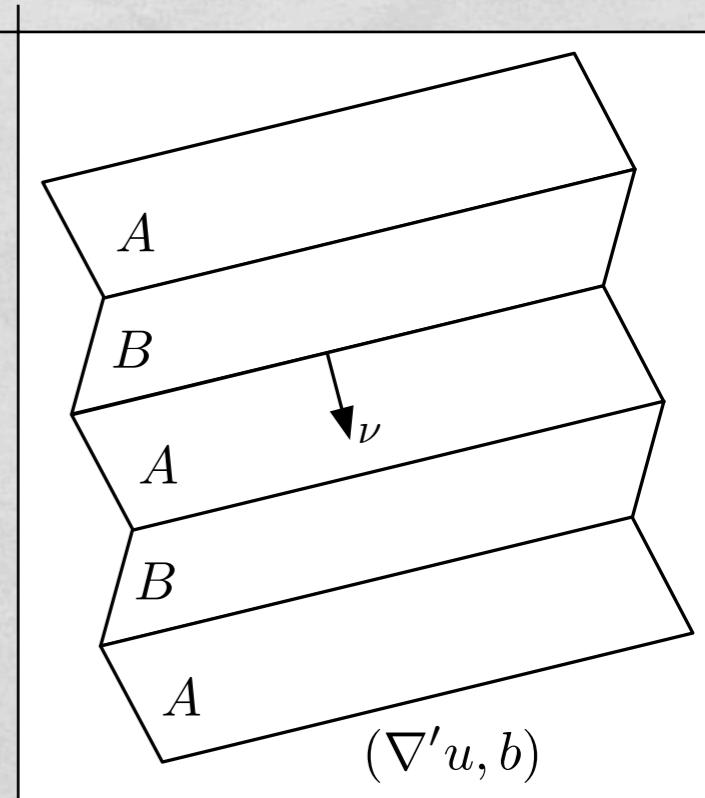
$u_\varepsilon \rightarrow u$
 $\frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b$

$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

$$\Rightarrow \begin{array}{|c|} \hline u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \nabla u \in \{(A'|0), (B'|0)\} \\ \hline \end{array}$$

Ball, James '87

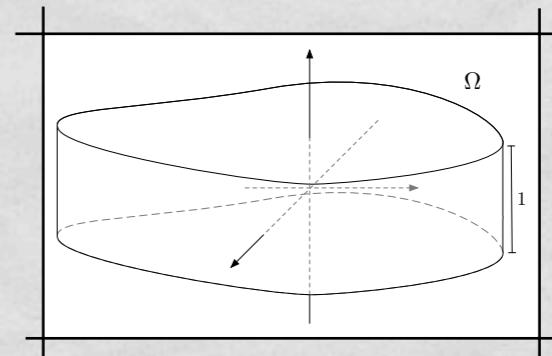
$$\Rightarrow A' - B' = a \otimes \nu \quad \text{rank-one connected}$$



PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

$$\begin{aligned} I_\varepsilon^1(u) := & \frac{1}{\varepsilon} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ & + \varepsilon \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx \end{aligned}$$

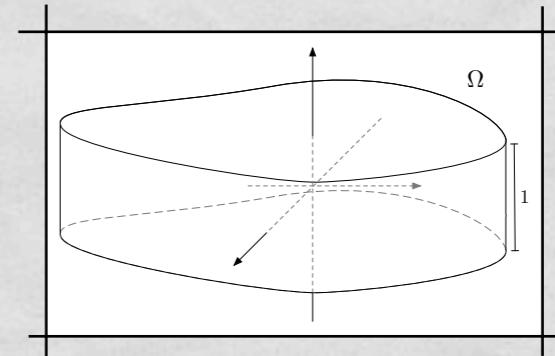


PHASE TRANSITIONS FOR THIN FILMS

$$\gamma = 1$$

Critical Case

$$I_\varepsilon^1(u) := \frac{1}{\varepsilon} \int_{\Omega} W\left(\nabla' u \left| \frac{1}{\varepsilon} \partial_3 u\right.\right) dx + \\ + \varepsilon \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



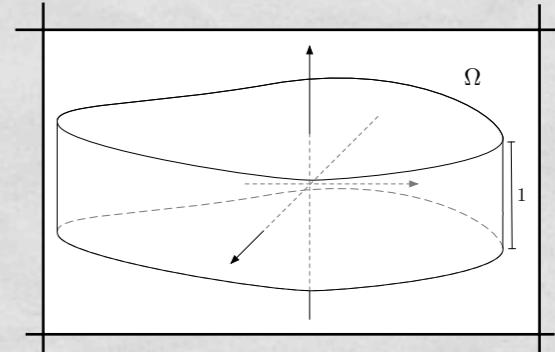
$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$$\gamma = 1$$

Critical Case

$$I_\varepsilon^1(u) := \frac{1}{\varepsilon} \int_{\Omega} W\left(\nabla' u \left\| \frac{1}{\varepsilon} \partial_3 u\right\|\right) dx + \\ + \varepsilon \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

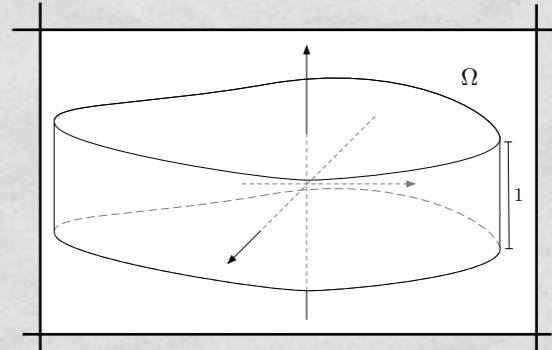


$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. u_\varepsilon \rightarrow u \right. \\ \left. \frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b \right. \\ \left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

$$I_\varepsilon^1(u) := \frac{1}{\varepsilon} \int_{\Omega} W\left(\nabla' u \left\| \frac{1}{\varepsilon} \partial_3 u\right\|\right) dx + \\ + \varepsilon \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^1(u, b) = \begin{cases} K_\nu^\star \operatorname{Per}_\omega(\{(\nabla' u|b) = A\}) & \text{if } (u, b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

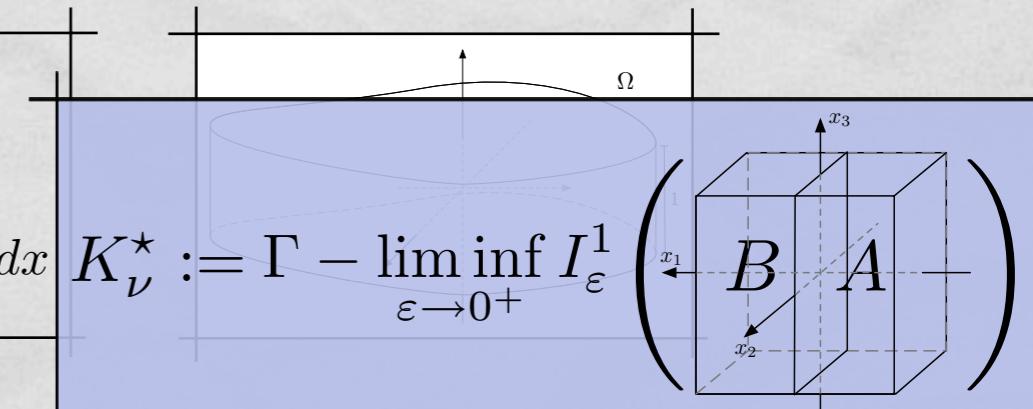
$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. \begin{array}{l} u_\varepsilon \rightarrow u \\ \frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b \end{array} \right\} \quad (\nabla' u|b) \in BV(\Omega; \{A, B\})$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

$$I_\varepsilon^1(u) := \frac{1}{\varepsilon} \int_{\Omega} W\left(\nabla' u \left\| \frac{1}{\varepsilon} \partial_3 u\right\|\right) dx +$$

$$+ \varepsilon \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^1(u, b) = \begin{cases} K_\nu^\star \text{Per}_\omega(\{(\nabla' u|b) = A\}) & \text{if } (u, b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

$$u_\varepsilon \rightarrow u$$

$$\frac{1}{\varepsilon} \partial_3 u_\varepsilon \rightarrow b$$

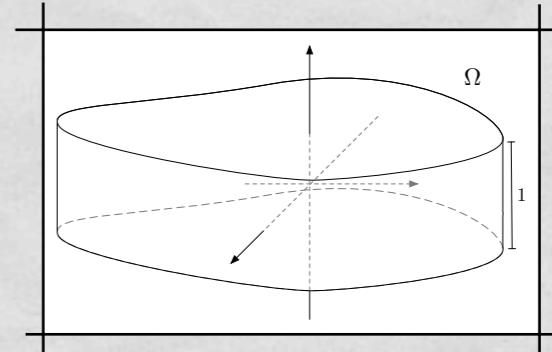
$$\left. (\nabla' u|b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$$\gamma < 1$$

Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

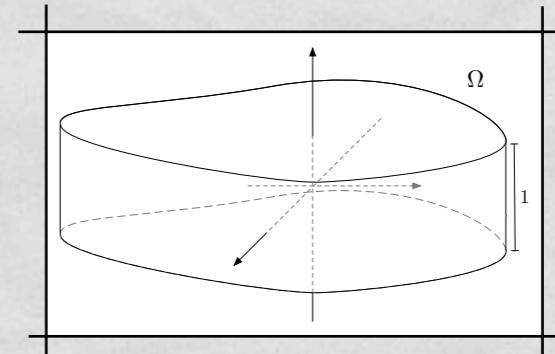


PHASE TRANSITIONS FOR THIN FILMS

$$\gamma < 1$$

Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

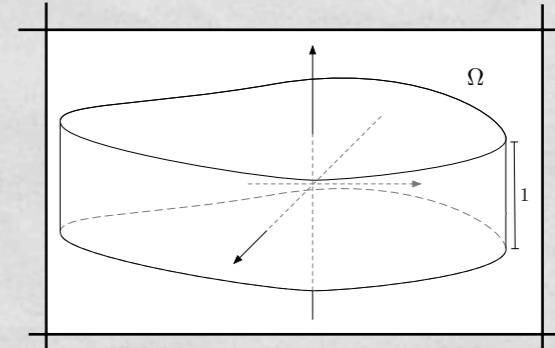


$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma < 1$ Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



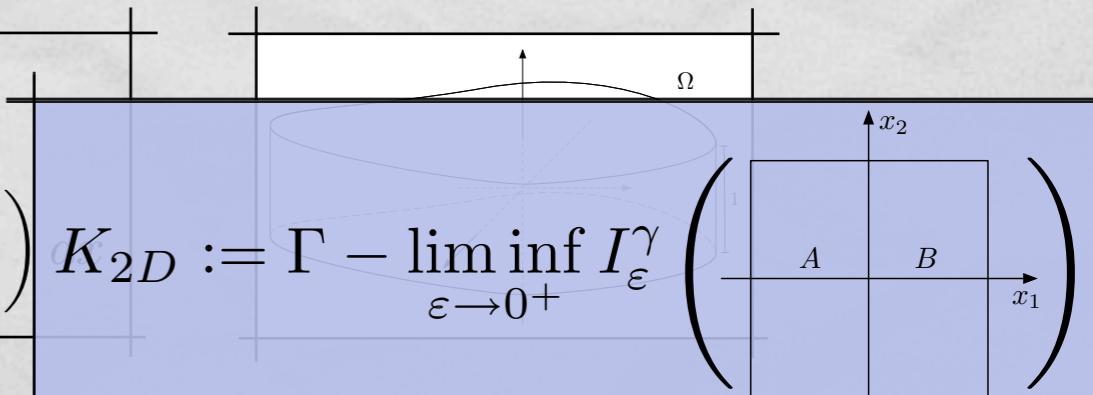
$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^\gamma(u, b) = \begin{cases} K_{2D} \operatorname{Per}_\omega \left(\{(\nabla' u | b) = A\} \right) & \text{if } (u, b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma < 1$ Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right)$$



$$K_{2D} := \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^\gamma \left(\cdot \middle| \begin{array}{c} \Omega \\ \text{cylinder} \\ A \quad B \end{array} \right)$$

$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^\gamma(u, b) = \begin{cases} K_{2D} \operatorname{Per}_\omega \left(\{(\nabla' u | b) = A\} \right) & \text{if } (u, b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

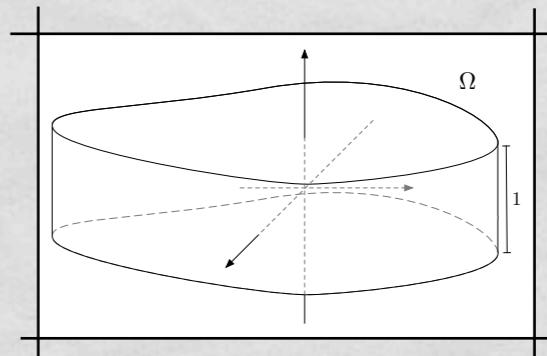
$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma > 1$ Supercritical Case

$A - B = a \otimes \nu$ rank-1 connected, $A' \neq B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

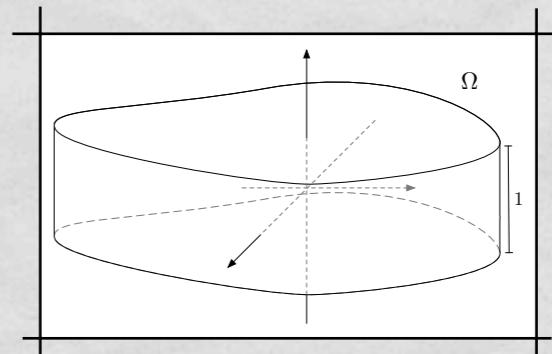


PHASE TRANSITIONS FOR THIN FILMS

$\gamma > 1$ Supercritical Case

$A - B = a \otimes \nu$ rank-1 connected, $A' \neq B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



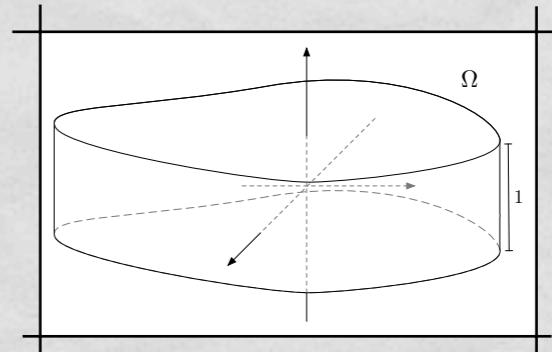
$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

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$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^\gamma(u, b) = \begin{cases} K^\gamma \operatorname{Per}_\omega \left(\{(\nabla' u | b) = A\} \right) & \text{if } (u, b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

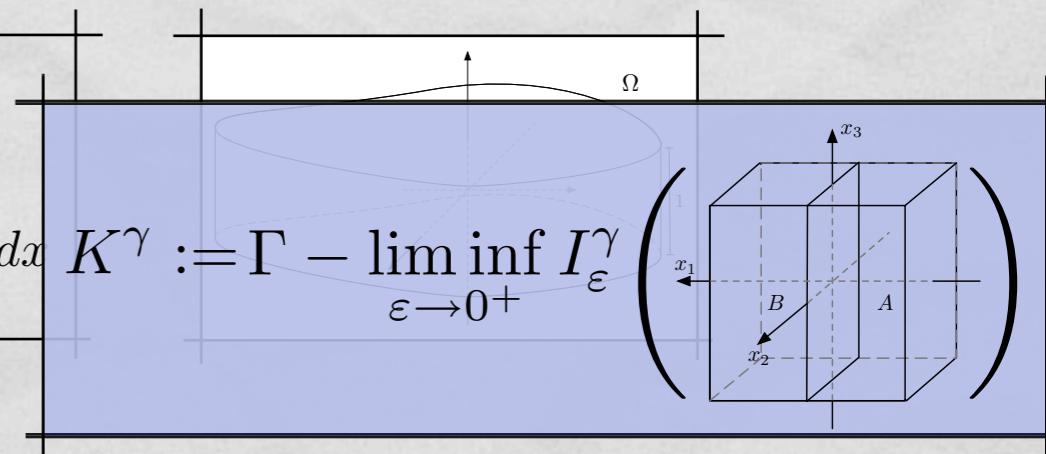
$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma > 1$ Supercritical Case

$A - B = a \otimes \nu$ rank-1 connected, $A' \neq B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^\gamma(u, b) = \begin{cases} \textcolor{blue}{K^\gamma} \operatorname{Per}_\omega \left(\{(\nabla' u | b) = A\} \right) & \text{if } (u, b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right.$$

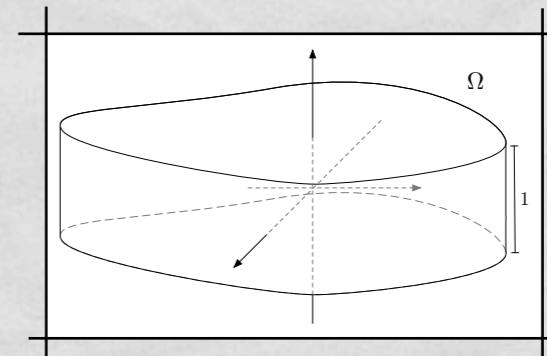
$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma > 1$ Supercritical Case

$\gamma > p, A' = B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$

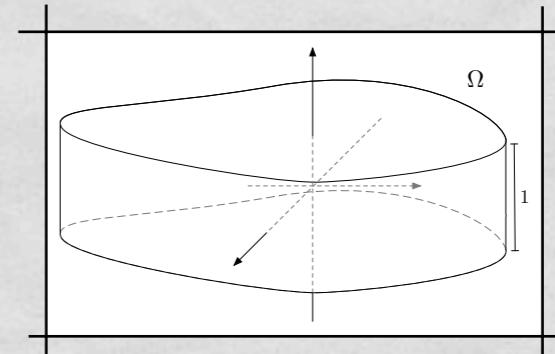


PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma > 1}$ Supercritical Case

$\gamma > p, A' = B'$

$$\begin{aligned} I_\varepsilon^\gamma(u) := & \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ & + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx \end{aligned}$$



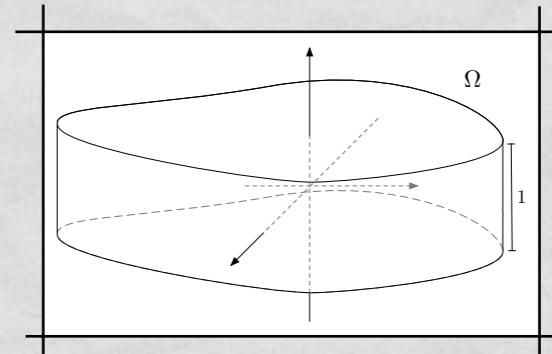
$$\boxed{\bar{\mathcal{V}} := \{(u, b) : \nabla u = (A', 0), b \in \{A_3, B_3\}\}}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma > 1}$ Supercritical Case

$\gamma > p, A' = B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_{\Omega} W\left(\nabla' u | \frac{1}{\varepsilon} \partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left| \frac{1}{\varepsilon} \nabla' (\partial_3 u) \right|^2 + \frac{1}{\varepsilon^2} \left| \frac{1}{\varepsilon} \partial_3^2 u \right|^2 \right) dx$$



$$\Gamma - \lim_{\varepsilon \rightarrow 0^+} I_\varepsilon^\gamma(u, b) = \begin{cases} 0 & \text{if } (u, b) \in \bar{\mathcal{V}}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\bar{\mathcal{V}} := \{(u, b) : \nabla u = (A', 0), b \in \{A_3, B_3\}\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step I. Γ - lim inf

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step I. Γ - lim inf

Scaling Properties

$$\Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\text{Diagram A} \right) = \alpha \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\text{Diagram B} \right)$$

The equation illustrates the scaling properties of the functional I_ε^1 .
Diagram A shows a rectangular prism with dimensions x_1 , x_2 , and x_3 . It is divided into two regions, B (bottom-left) and A (top-right), by a vertical plane. A red double-headed arrow labeled α indicates a scaling factor applied to the entire volume.
Diagram B shows a similar rectangular prism with dimensions x_1 , x_2 , and x_3 . It is also divided into regions B and A by a vertical plane. A red double-headed arrow labeled 1 indicates a scaling factor applied to the entire volume.

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step I. Γ - lim inf

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step I. Γ - lim inf

Energy Concentrates on Discontinuity Surface

$$\Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\text{Diagram A} \right) = \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\text{Diagram B} \right)$$

The equation shows two diagrams representing different configurations of a thin film across a unit cube of side length 1. In both diagrams, the cube is defined by axes x_1 , x_2 , and x_3 . A vertical plane labeled 'A' divides the cube. A diagonal plane labeled 'B' intersects plane 'A'. In Diagram A, the intersection of planes A and B is a single vertical line segment. In Diagram B, the intersection is a rectangular surface. The width of this surface is indicated by a red number '1' below Diagram A and a red symbol ' δ ' below Diagram B.

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$

Critical Case

Idea of Proof

Step I. Γ - lim inf

$$\Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left((u, b); \begin{array}{c} \Omega \\ \parallel \end{array} \right) \geq K_\nu^\star \mathcal{H}^1(S(\nabla' u, b))$$

$$K_\nu^\star := \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\begin{array}{c} x_3 \\ x_1 \quad B \quad A \\ x_2 \end{array} \right)$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$

$$A = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right)$$

A' A_3

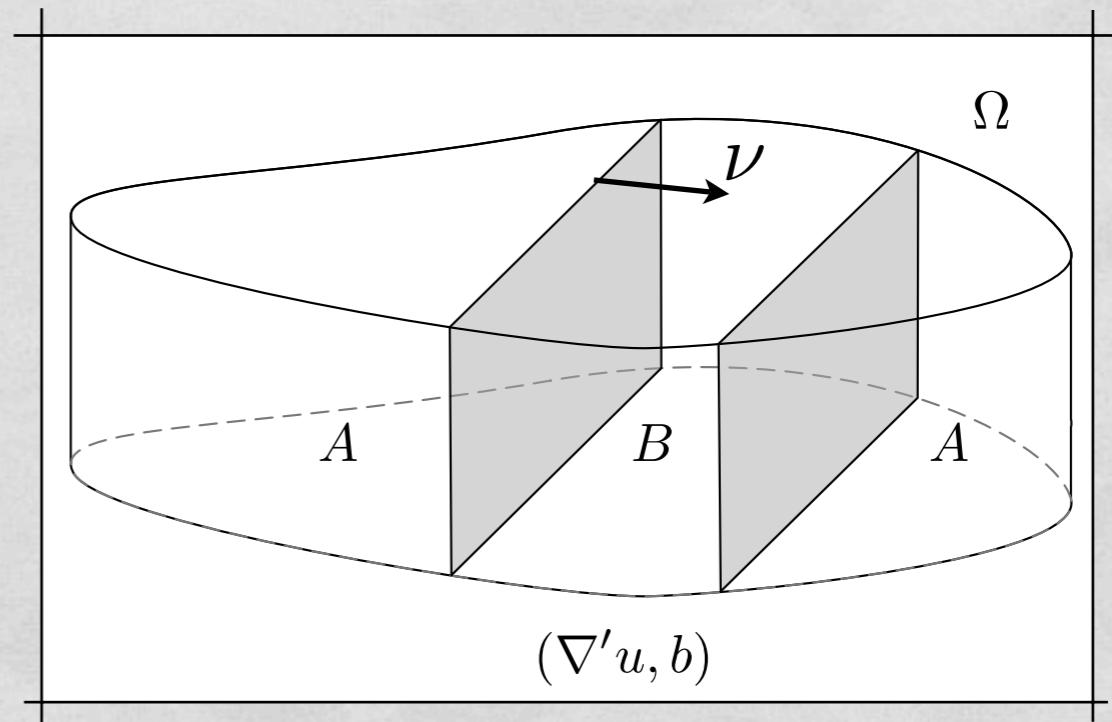
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$



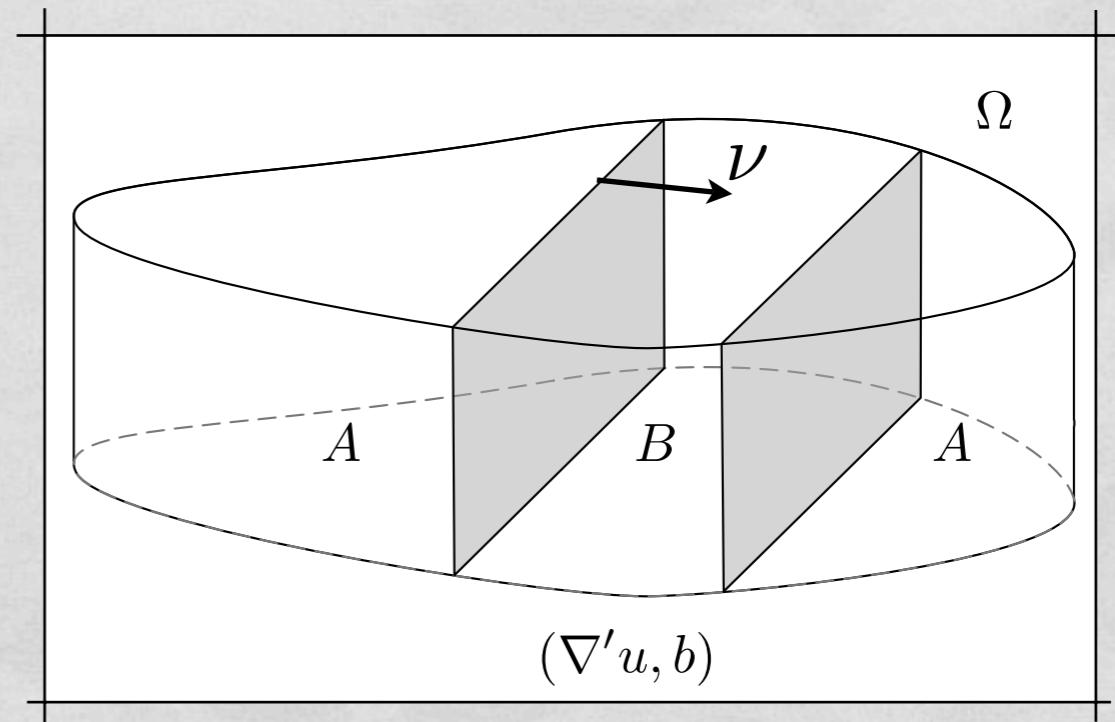
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$



PHASE TRANSITIONS FOR THIN FILMS

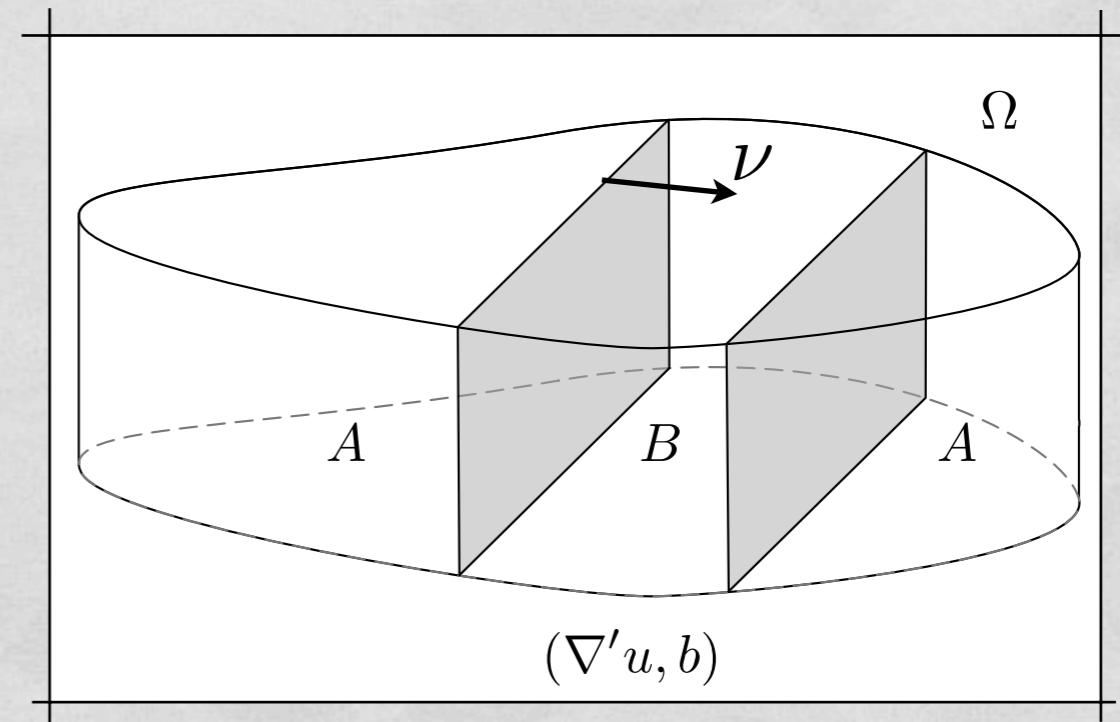
$\gamma = 1$

Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$



(H_4) best path from A' to B' does not depend on ν^\perp

PHASE TRANSITIONS FOR THIN FILMS

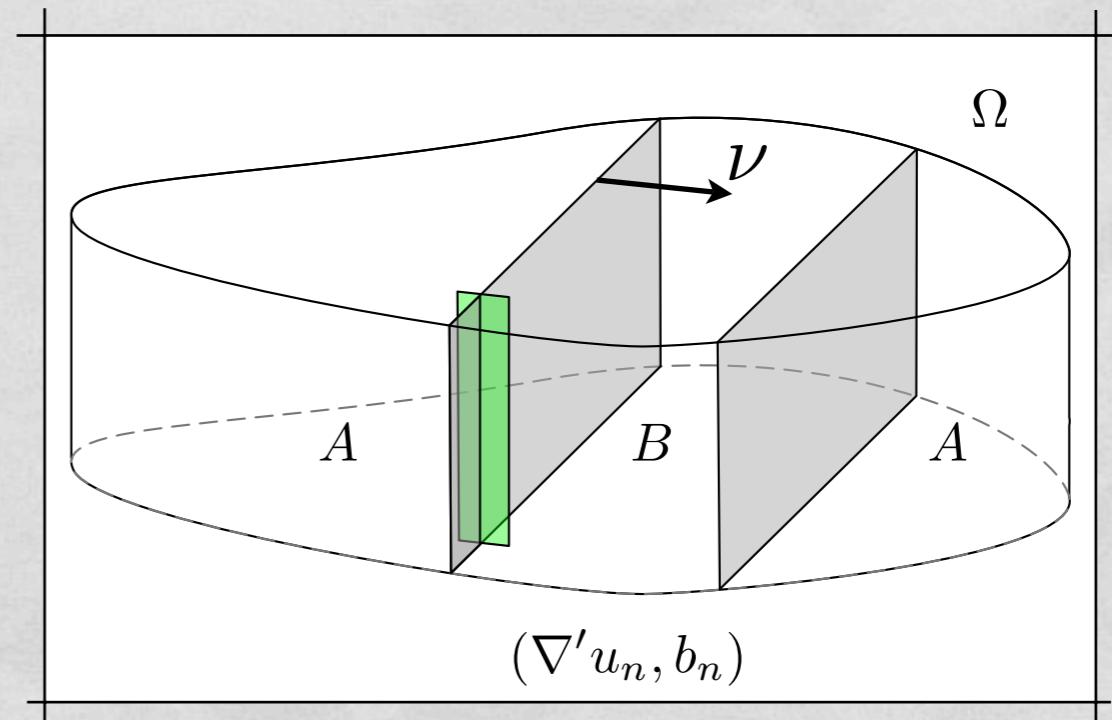
$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$

Construct
Recovery Sequence



(H_4) best path from A' to B' does not depend on ν^\perp

PHASE TRANSITIONS FOR THIN FILMS

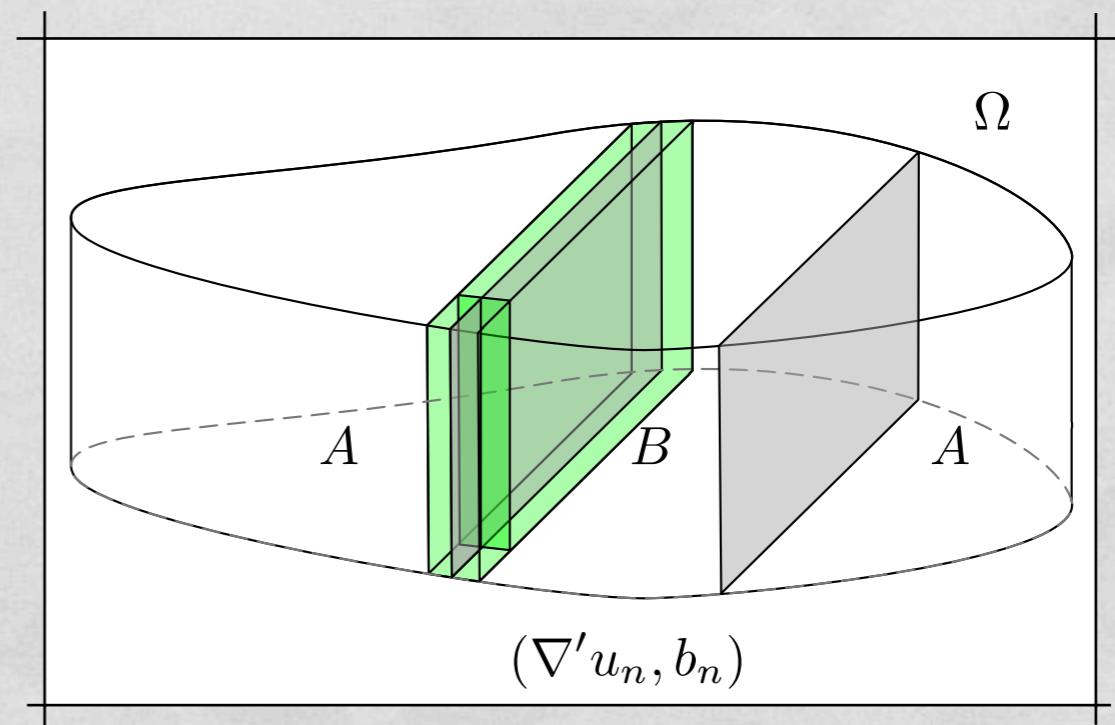
$\gamma = 1$

Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$



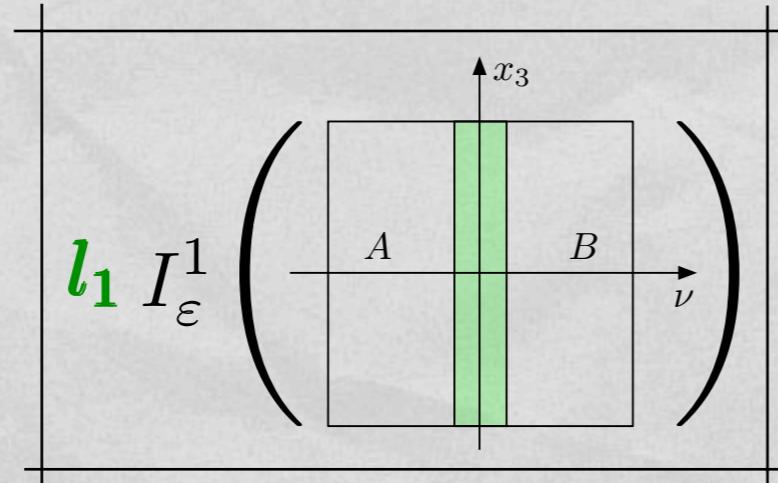
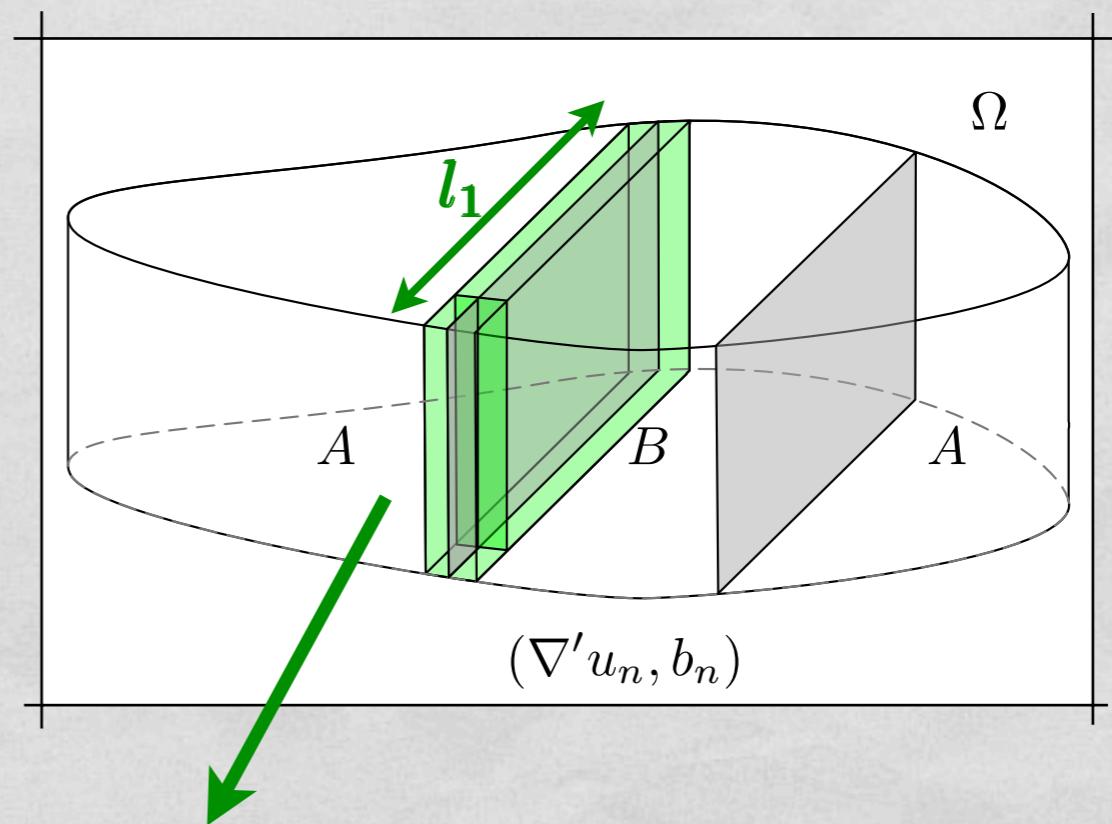
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$



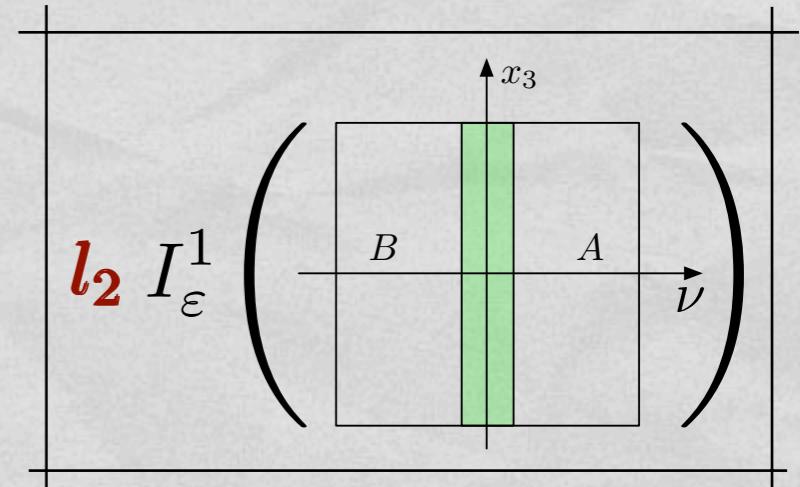
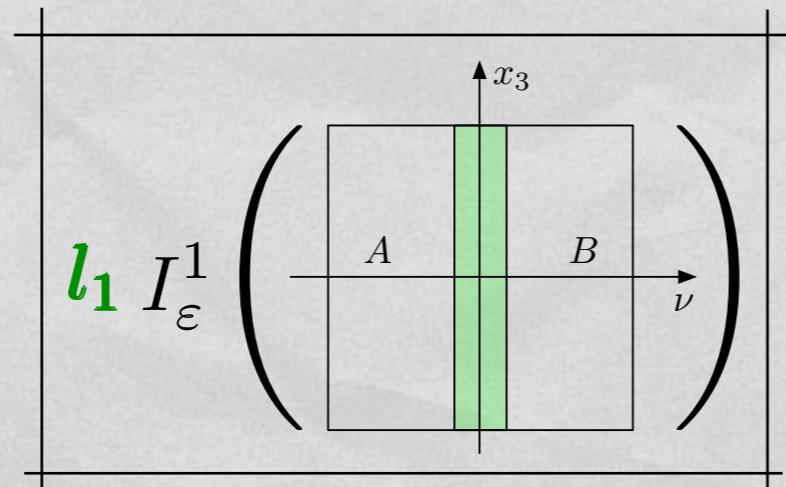
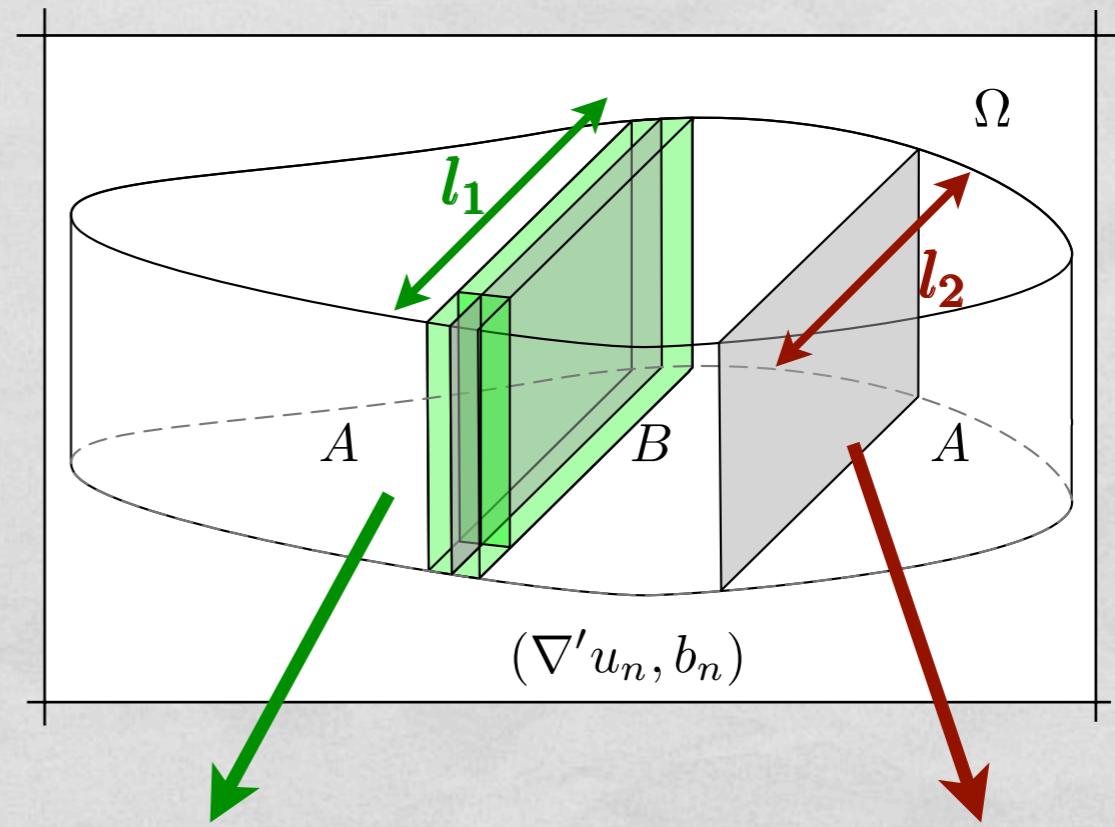
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$



PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$

$$I_\varepsilon^1 \left(u_\varepsilon; \Omega \right) \leq I_\varepsilon^1 \left(\text{Diagram} \right) \mathcal{H}^1 \left(S(\nabla' u, b) \right)$$

The diagram illustrates a thin film of thickness 1 over a domain Ω . The left side shows the physical domain Ω with a vertical axis and a horizontal boundary. The right side shows a cross-section of the film, with axes x_1 , x_2 , and x_3 . The domain is divided into regions B and A by a vertical interface. A green vertical strip represents the singular set $S(\nabla' u, b)$.

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$

Critical Case

Idea of Proof

Step 2. Γ - \limsup

Case 2a. $A' \neq B'$

$$I_\varepsilon^1 \left(u_\varepsilon; \begin{array}{c} \text{3D cylinder } \Omega \\ \text{with height } 1 \end{array} \right) \leqslant I_\varepsilon^1 \left(\begin{array}{c} \text{2D domain } \nu \\ \text{partitioned into } B \text{ and } A \\ \text{with a vertical strip } S(\nabla' u, b) \end{array} \right) \mathcal{H}^1(S(\nabla' u, b))$$

$$K_\nu^\star := \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\begin{array}{c} \text{2D domain } \nu \\ \text{partitioned into } B \text{ and } A \\ \text{with a vertical strip } S(\nabla' u, b) \end{array} \right)$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$

Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2a. $A' \neq B'$

$$I_\varepsilon^1 \left(u_\varepsilon; \begin{array}{c} \Omega \\ \text{---} \\ \text{---} \end{array} \right) \leq K_\nu^\star \mathcal{H}^1(S(\nabla' u, b))$$

$$K_\nu^\star := \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(\begin{array}{c} x_3 \\ \text{---} \\ \text{---} \\ B \quad A \\ \nu \end{array} \right)$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

A' A_3

PHASE TRANSITIONS FOR THIN FILMS

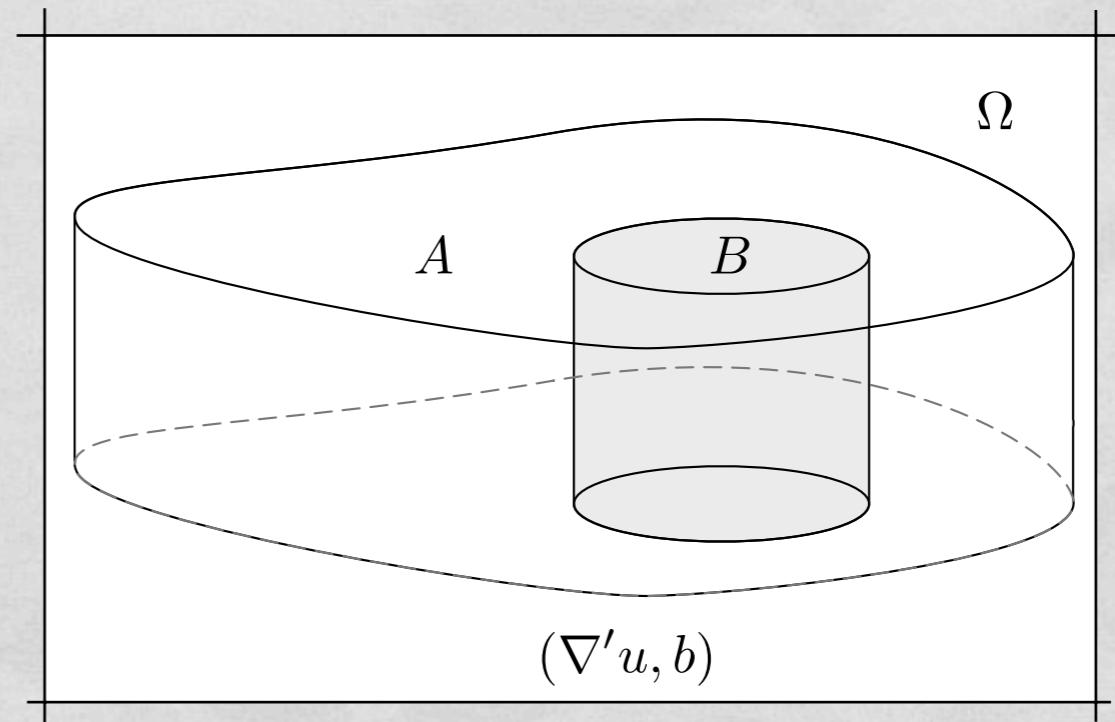
$\gamma = 1$

Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$



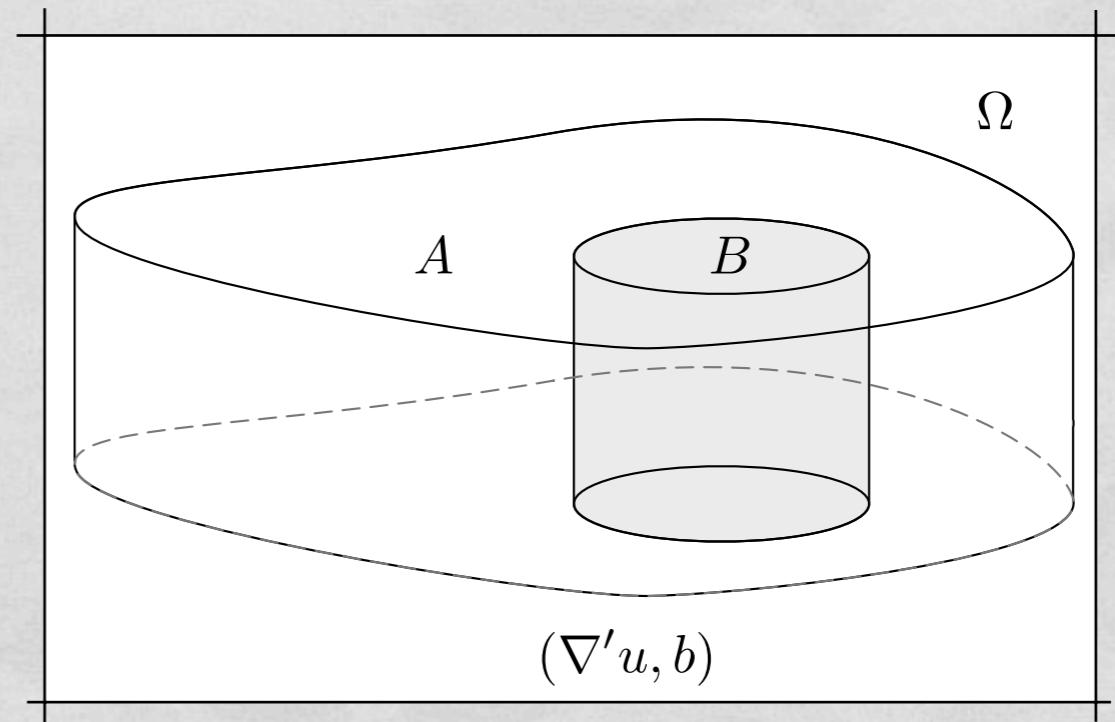
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$



$$(H_4) \quad W = W(|\xi'|, \xi_3)$$

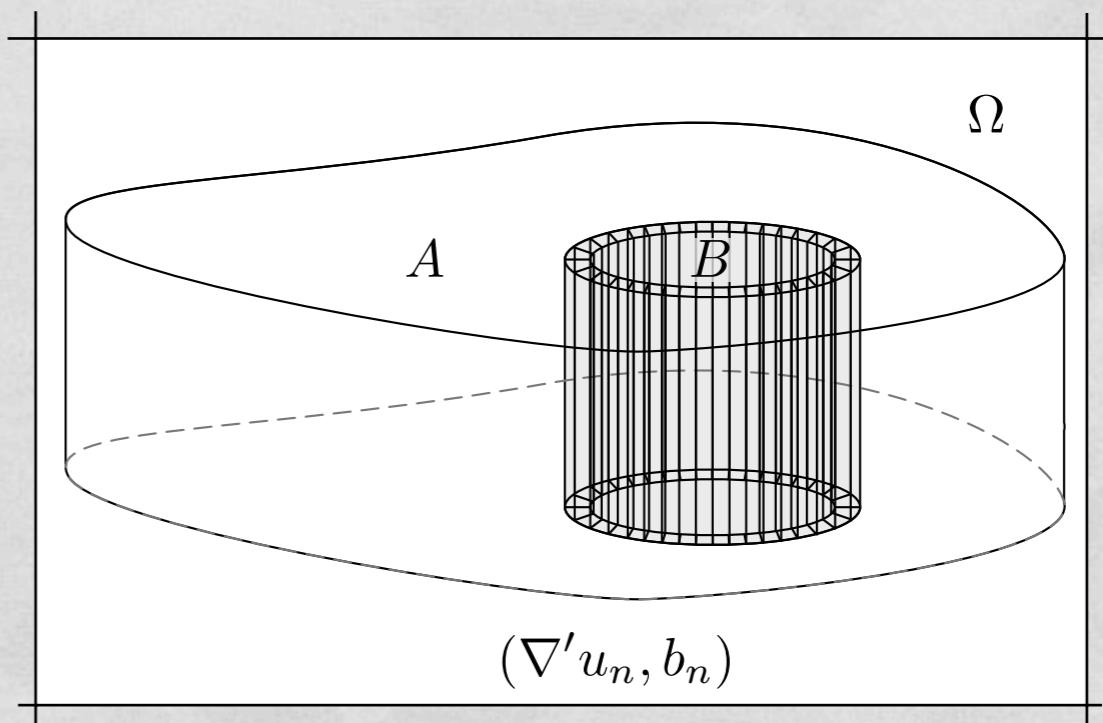
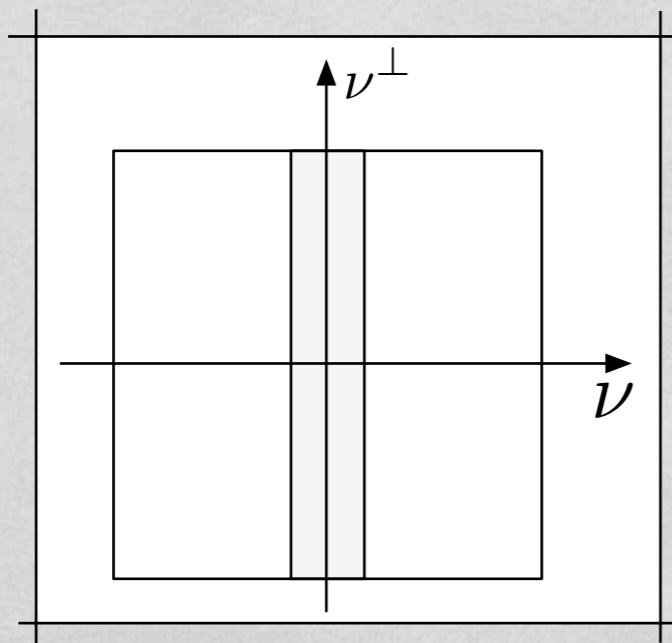
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$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$



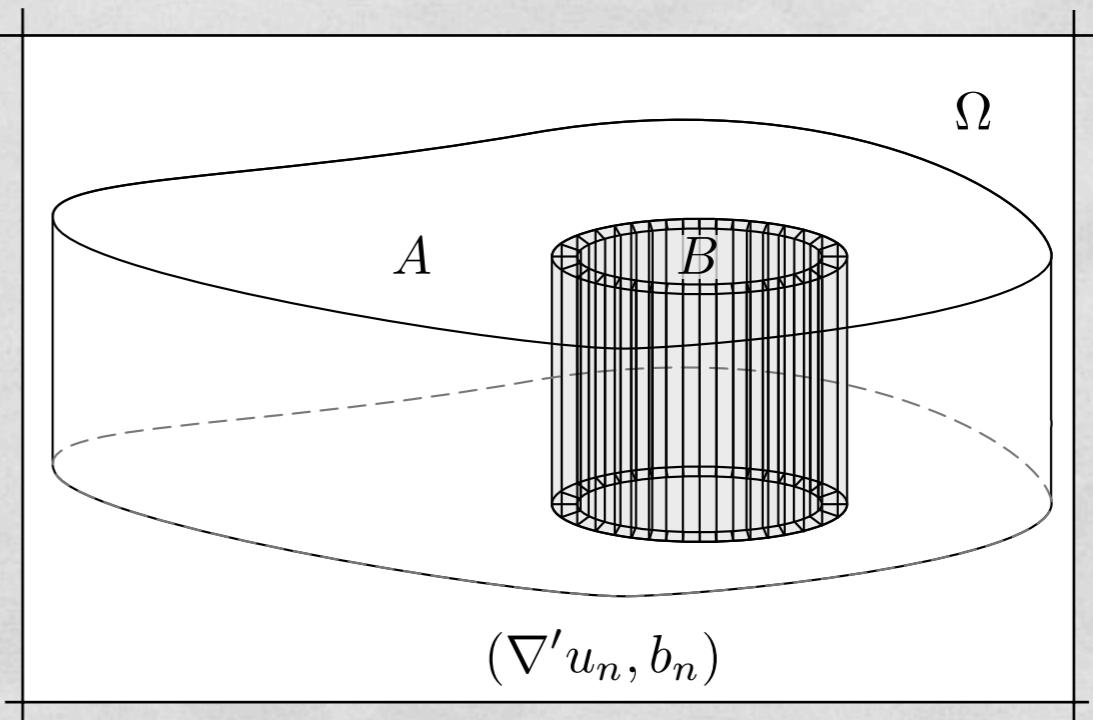
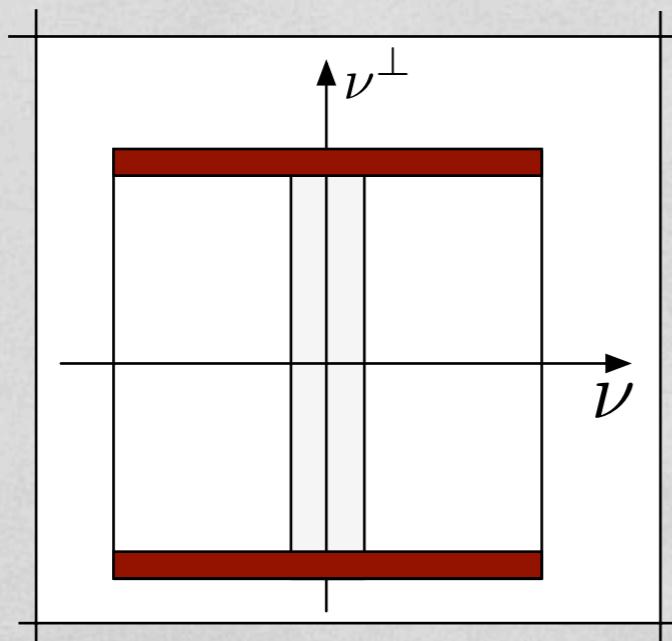
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$



Need to “glue” the cubes periodically on ν^\perp

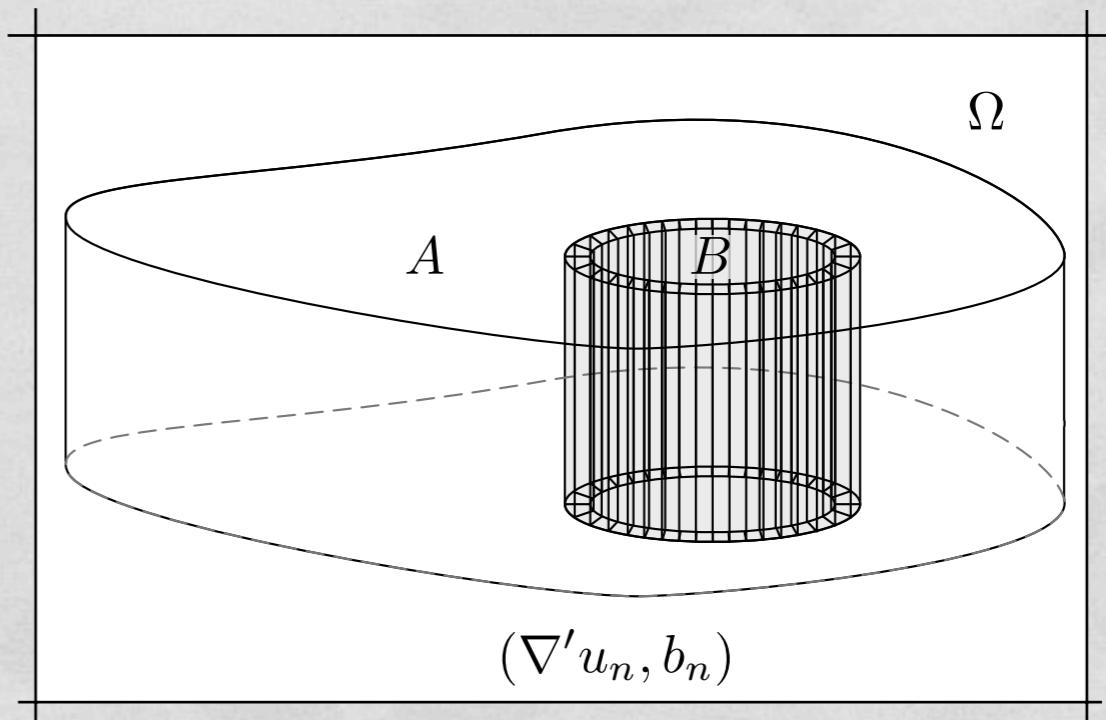
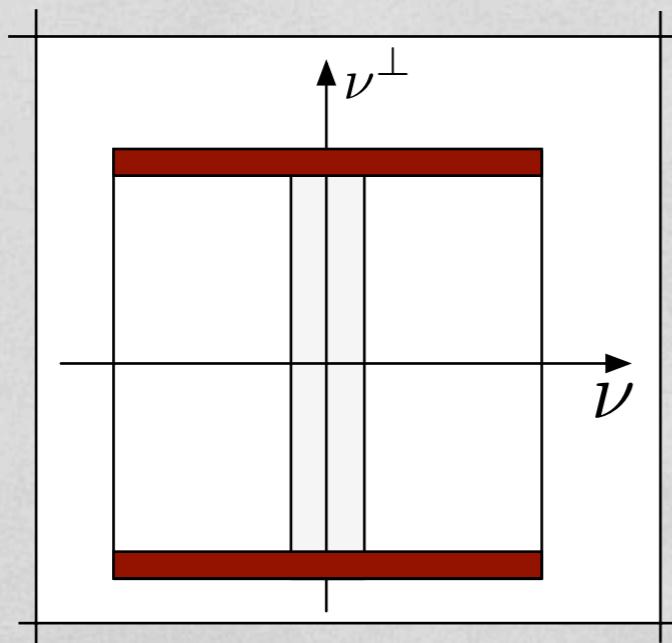
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$



Need to “glue” the cubes periodically on ν^\perp

Use Scaling Properties of Energy

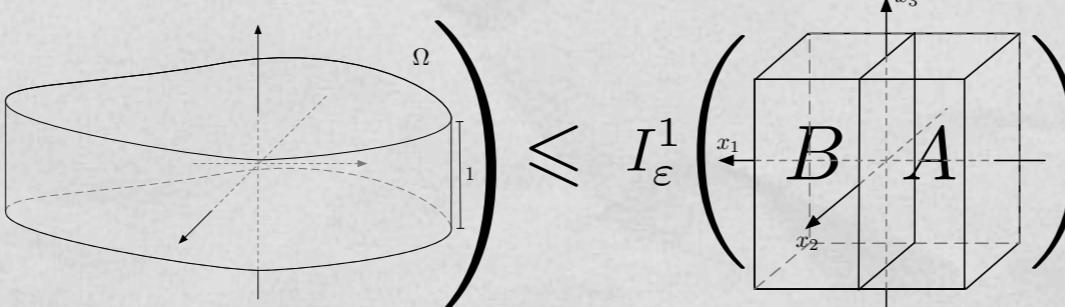
PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$$I_\varepsilon^1 \left((u, b); \Omega \right) \leq I_\varepsilon^1 \left(B \cup A \right) \mathcal{H}^1 \left(S(\nabla' u, b) \right)$$


The diagram consists of two parts. On the left, there is a 2D sketch of an elliptical domain Ω centered at the origin of a coordinate system, with a vertical axis labeled 1 indicating its height. On the right, there is a 3D sketch of a unit cube with vertices at $(0,0,0)$ and $(1,1,1)$. The cube is divided into two regions: B , which is the lower-left portion of the cube, and A , which is the upper-right portion. The axes are labeled x_1 , x_2 , and x_3 .

PHASE TRANSITIONS FOR THIN FILMS

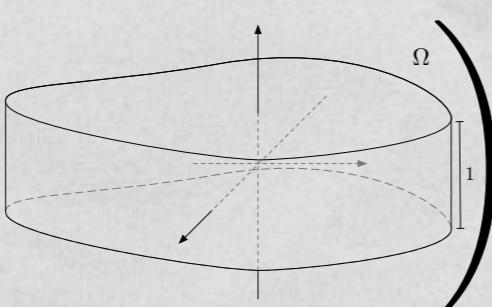
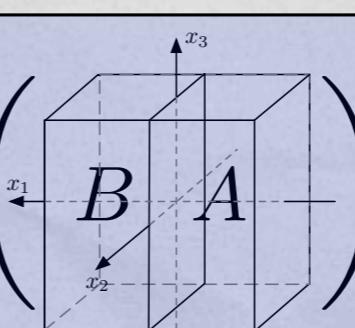
$\gamma = 1$

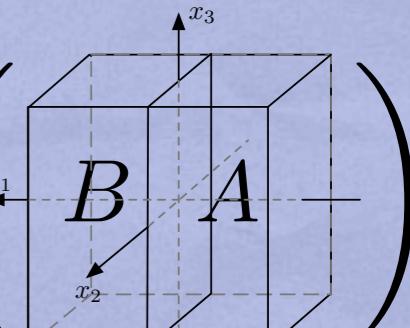
Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$$I_\varepsilon^1 \left((u, b); \Omega \right) \leq I_\varepsilon^1 \left(B \cup A; \mathbb{R}^3 \right) + \mathcal{H}^1 \left(S(\nabla' u, b) \right)$$



$$K_\nu^\star := \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(B \cup A; \mathbb{R}^3 \right)$$


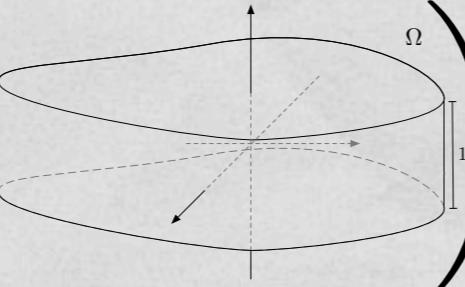
PHASE TRANSITIONS FOR THIN FILMS

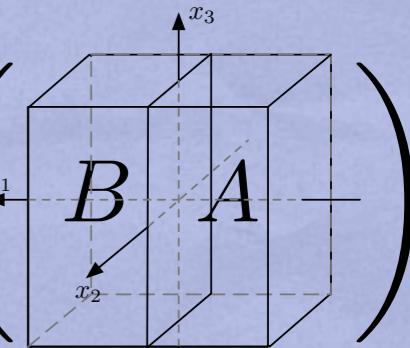
$\gamma = 1$ Critical Case

Idea of Proof

Step 2. Γ - lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$$I_\varepsilon^1 \left((u, b); \Omega \right) \leq K_\nu^\star \mathcal{H}^1(S(\nabla' u, b))$$


$$K_\nu^\star := \Gamma - \liminf_{\varepsilon \rightarrow 0^+} I_\varepsilon^1 \left(B \cup A \right)$$


PHASE TRANSITIONS FOR THIN FILMS

PHASE TRANSITIONS FOR THIN FILMS

Thank You

PHASE TRANSITIONS IN THIN FILMS

Outline of the Talk

Historic Context

for Phase Transitions

for Thin Films

Γ - limit

Phase Transitions in Thin Films

Hypotheses on W

Different Regimes

Rigidity

Critical Case

Subcritical Case

Supercritical Case

Γ - lim inf

Γ - lim sup

Case $A' \neq B'$

Case $A' = B', A_3 \neq B_3$