Phase Transitions for Thin Films

Bernardo Galvão-Sousa
Carnegie Mellon University

joint work with Vincent Millot

HISTORIC CONTEXT

classical theory for phase transitions for non-interacting fluid

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classical theory for phase transitions for non-interacting fluid

Ω container

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classical theory for phase transitions for non-interacting fluid

Ω container

u fluid density satisfies

$$\boxed{\text{mass: } \int_\Omega u \, dx = V}$$

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classical theory for phase transitions for non-interacting fluid

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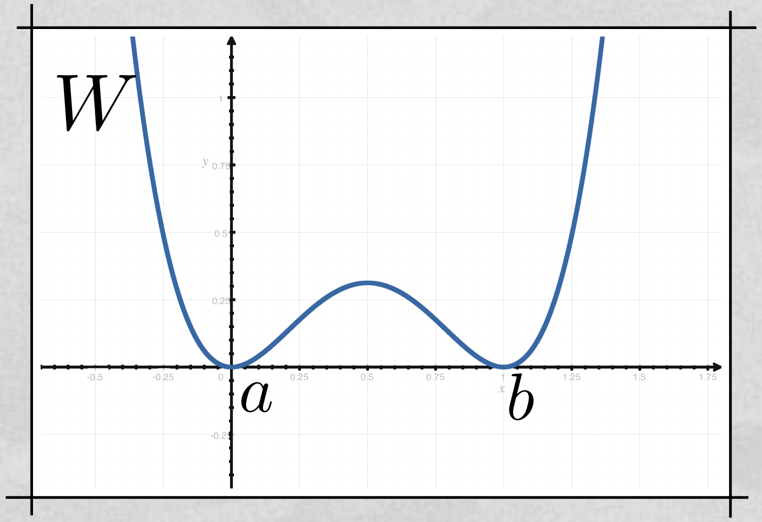
$$\boxed{\begin{gathered}\min_{\substack{u\in L^1(\Omega)\\ \int_\Omega u\,dx = V}} \int_\Omega W(u)\,dx \\ a|\Omega| < V < b|\Omega|\end{gathered}}$$



W = Gibbs free energy density

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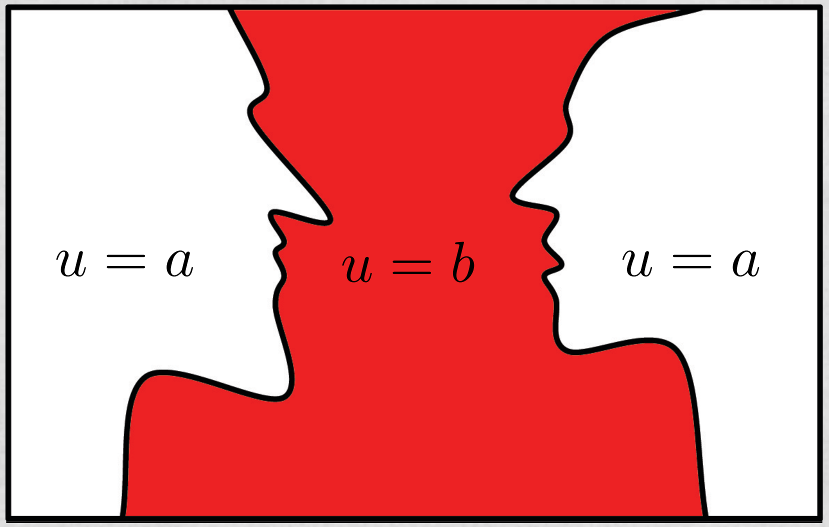
classical theory for phase transitions for non-interacting fluid



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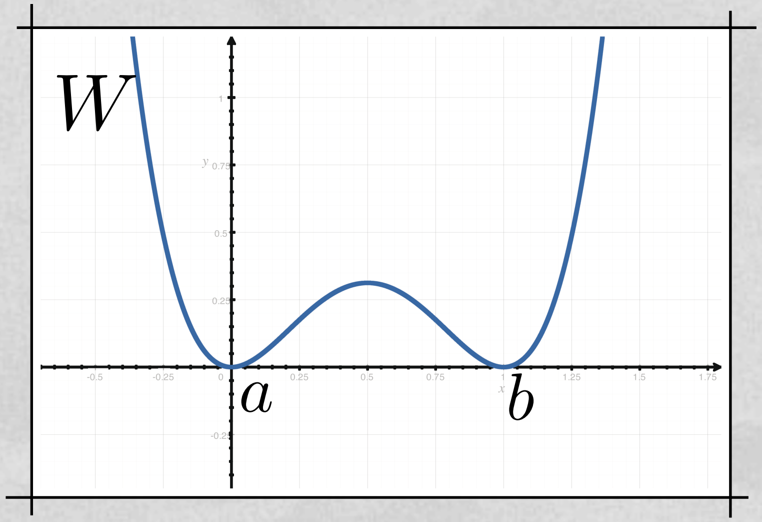

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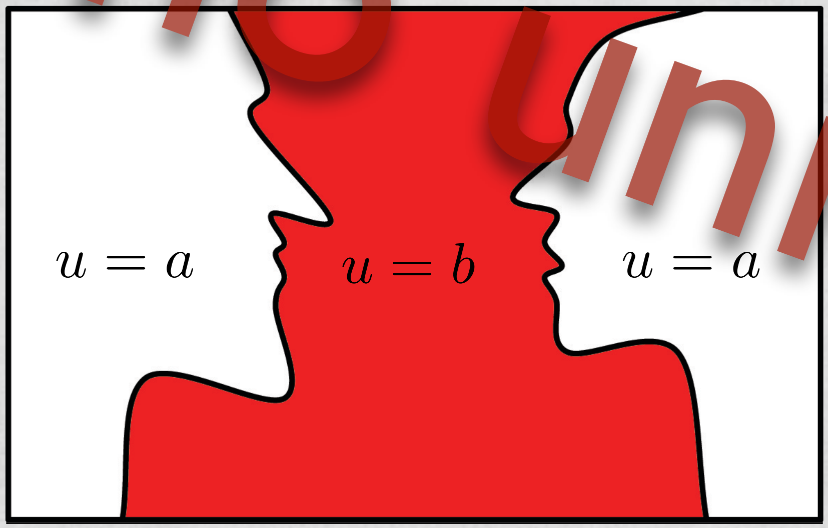
PHASE TRANSITIONS

classical theory for phase transitions for non-interacting fluid

no uniqueness



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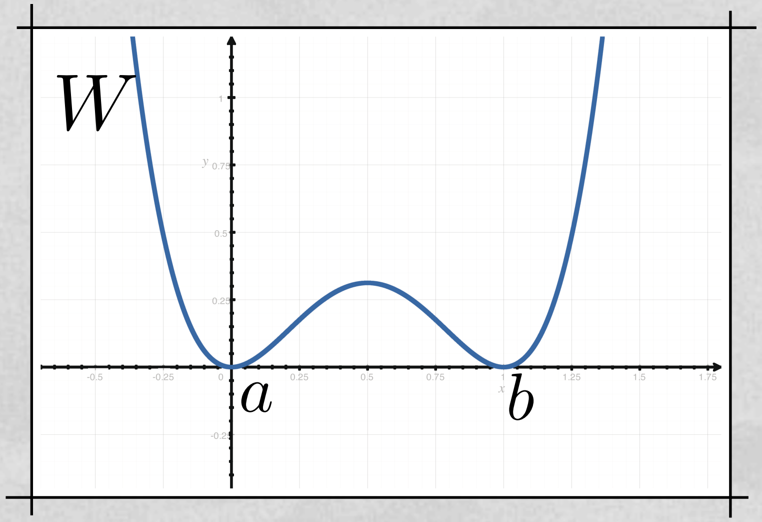

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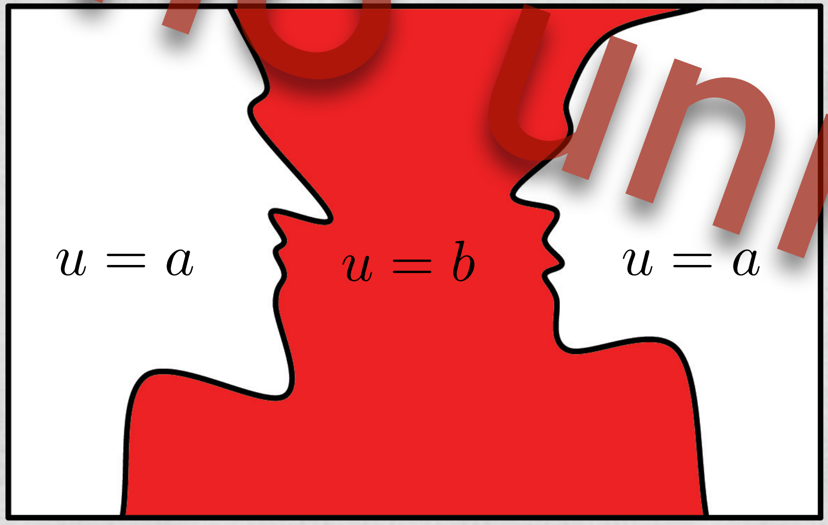

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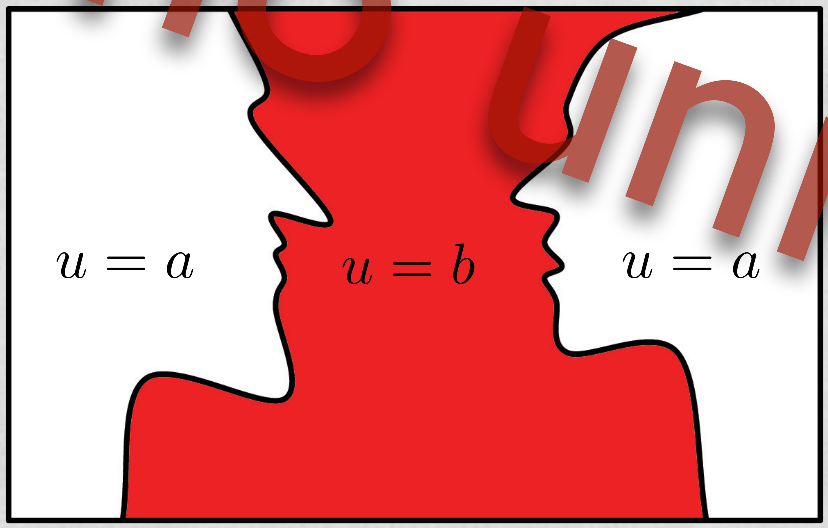

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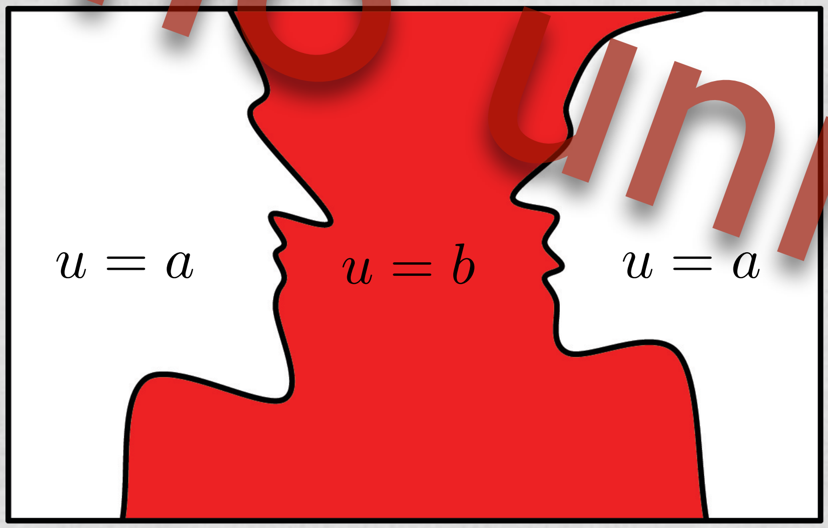

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no uniqueness

Van der Waals-Cahn-Hilliard

+ perturbation ⇨ selection criterium

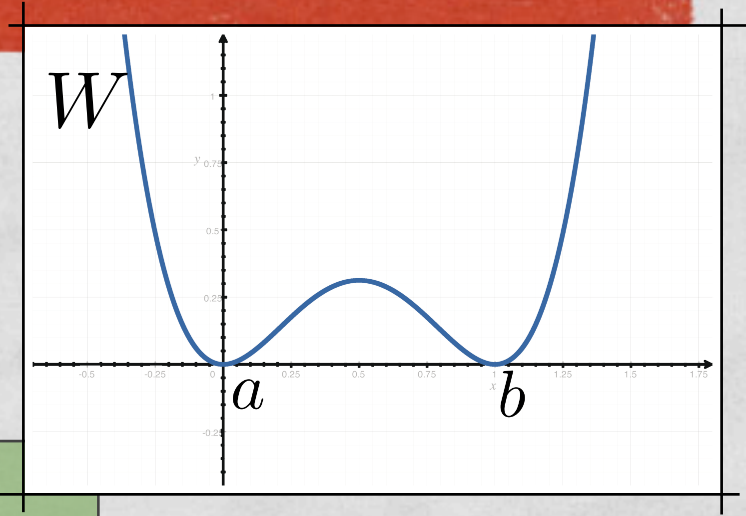
$$\min_{\substack{u \in L^1(\Omega) \\ \int_\Omega u\, dx = V}} \left\{ \int_\Omega W(u)\, dx + \varepsilon^{2k} \int_\Omega |D^k u|^2\, dx \right\}$$

HISTORIC CONTEXT

PHASE TRANSITIONS

Gurtin '85

$$\varepsilon \int_\Omega |Du|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx$$



minimizer u_ε

$$u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u_0$$

u_0 physically-preferred solution

minimizes interface area

$$\mathrm{Per}_\Omega(\{u_0 = a\}) = \min \mathrm{Per}_\Omega(\{u = a\})$$

conjecture

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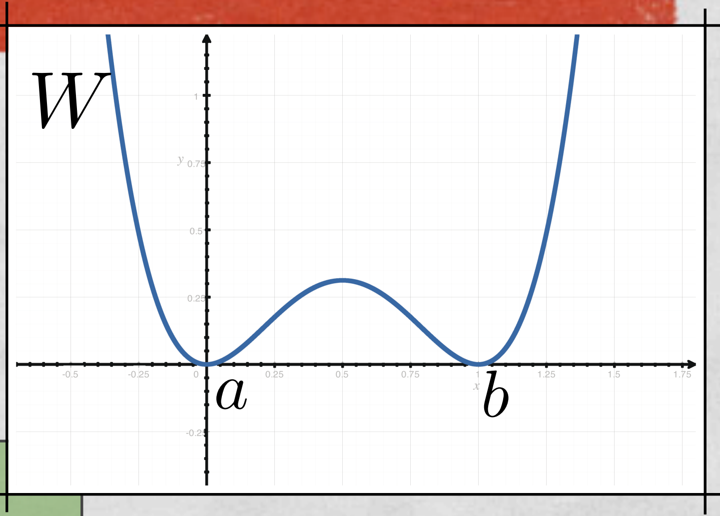

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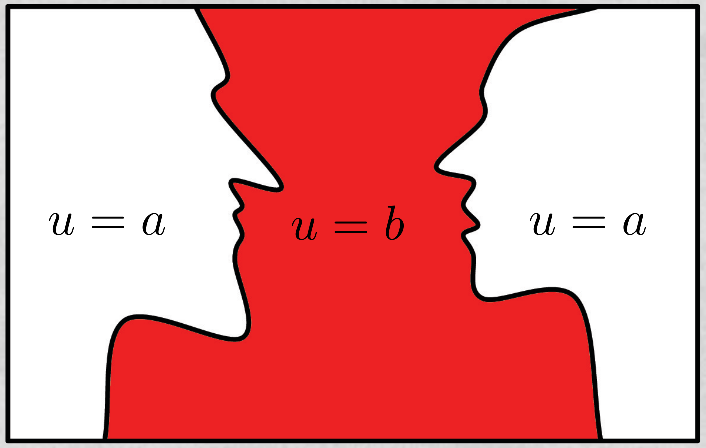

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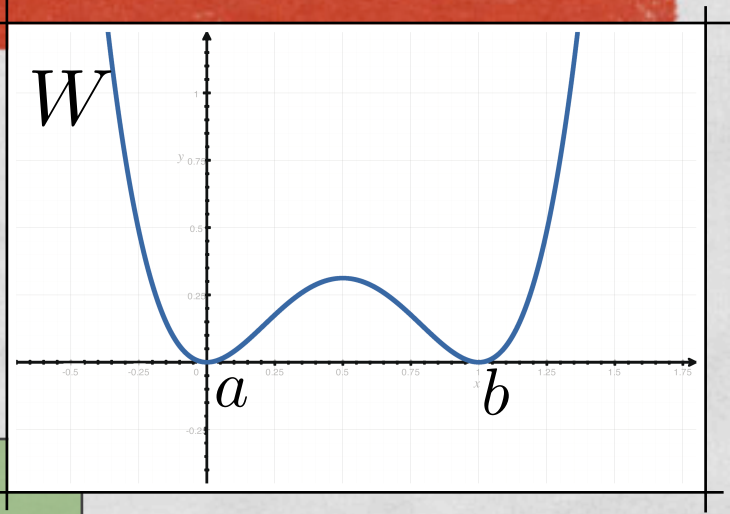

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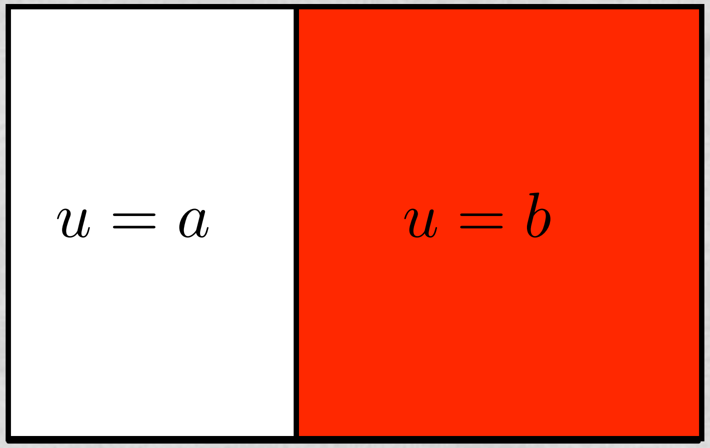

HISTORIC CONTEXT

Modica, Mortola '77

$$\boxed{\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\sin^2\left(\frac{\pi u}{\varepsilon}\right)\,dx+\varepsilon\int_{\mathbb{R}^N}|Du|^2\,dx}$$

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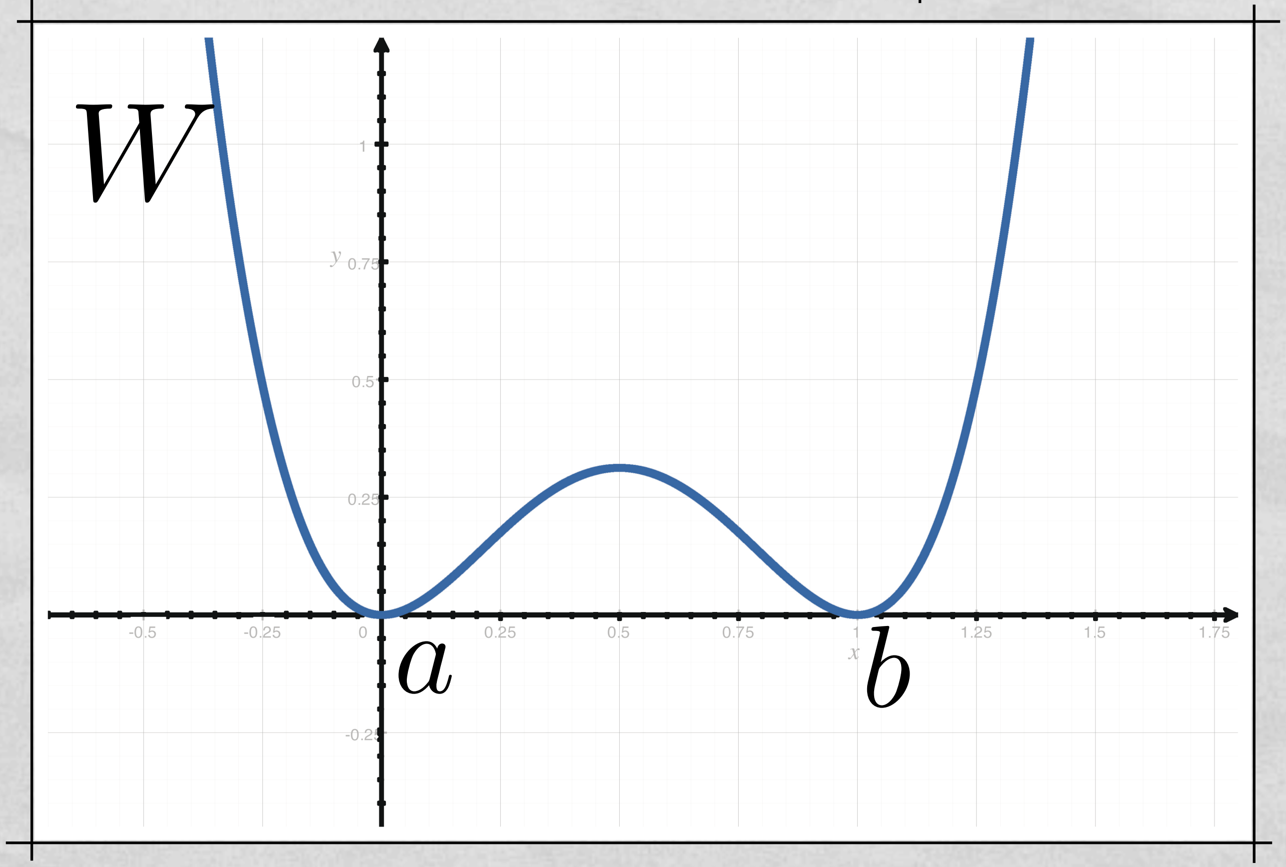

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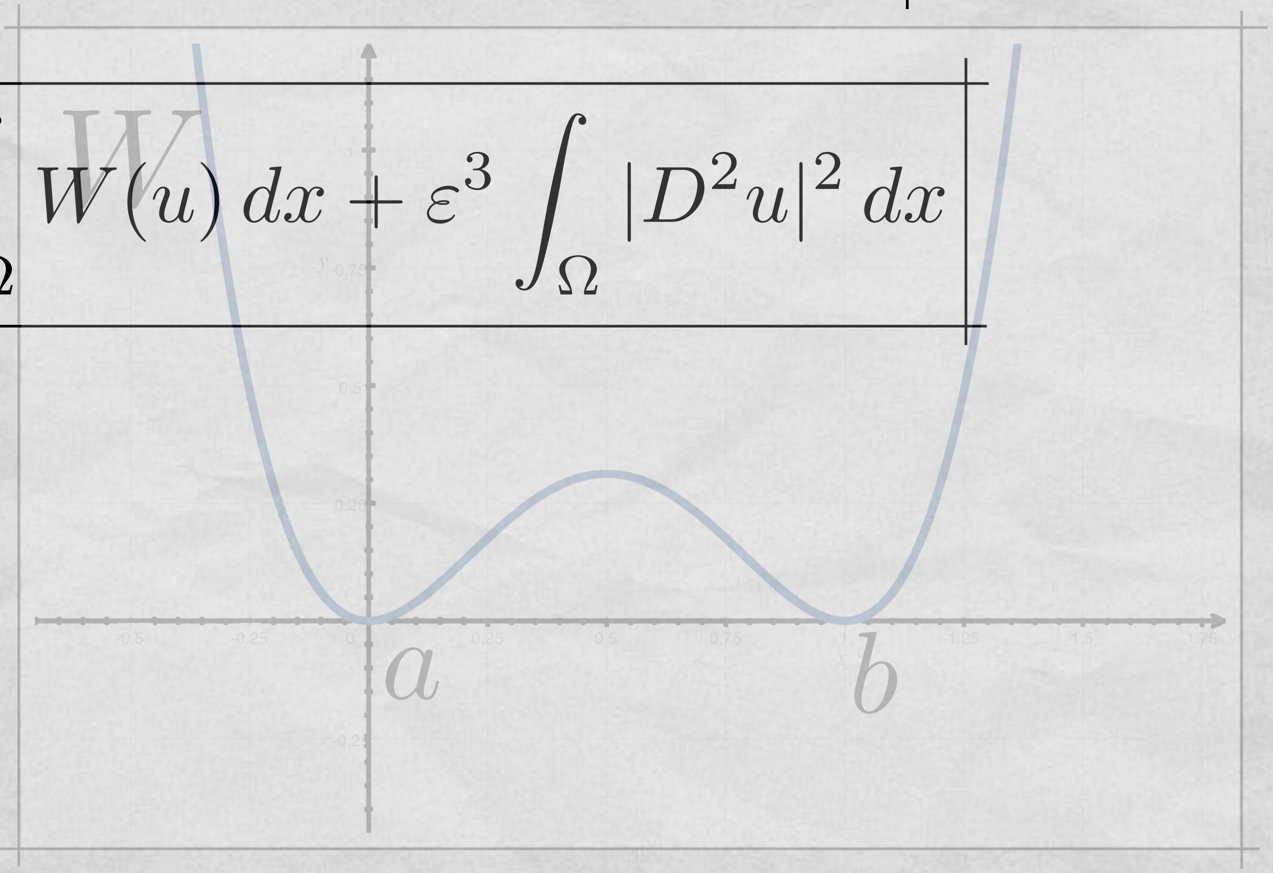

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Conti, Fonseca, Leoni ‘02

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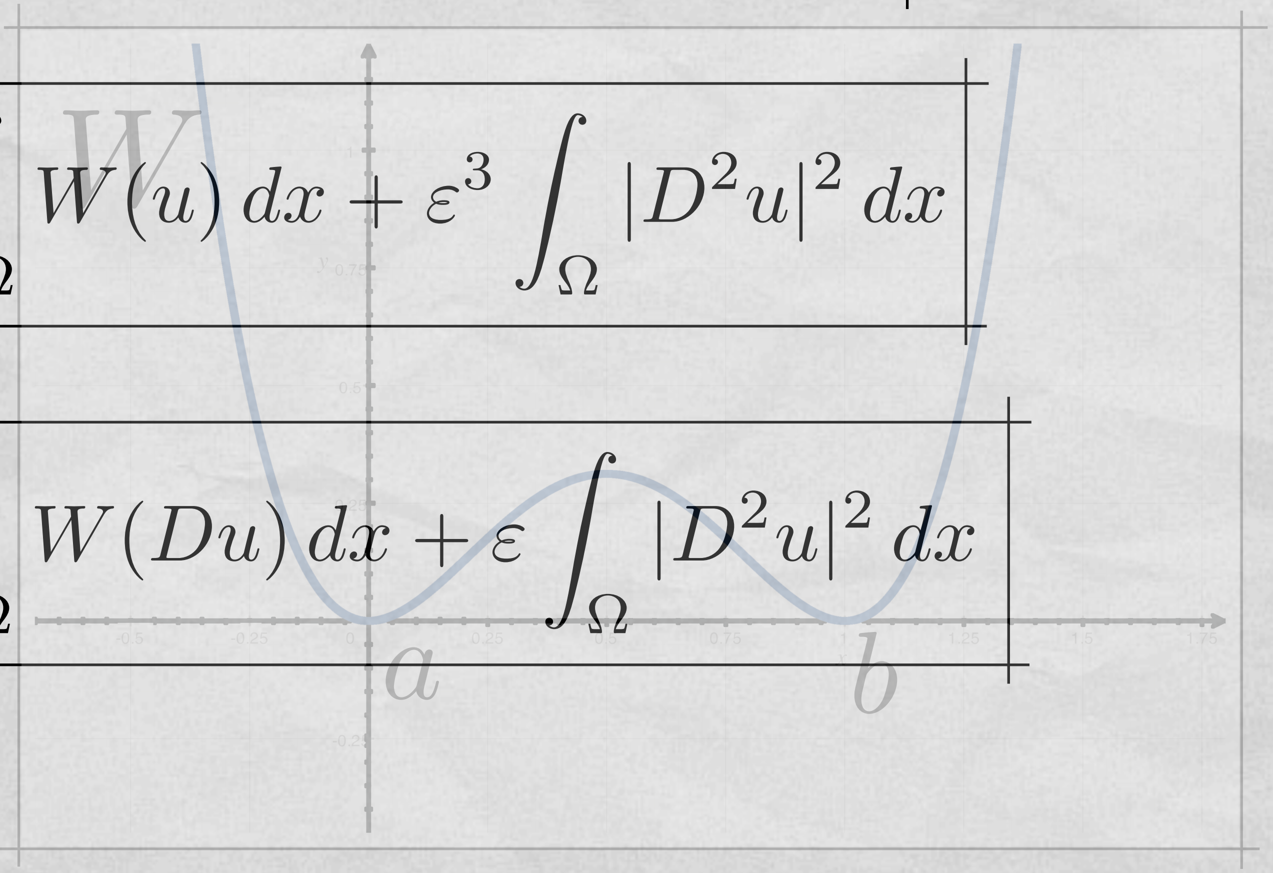

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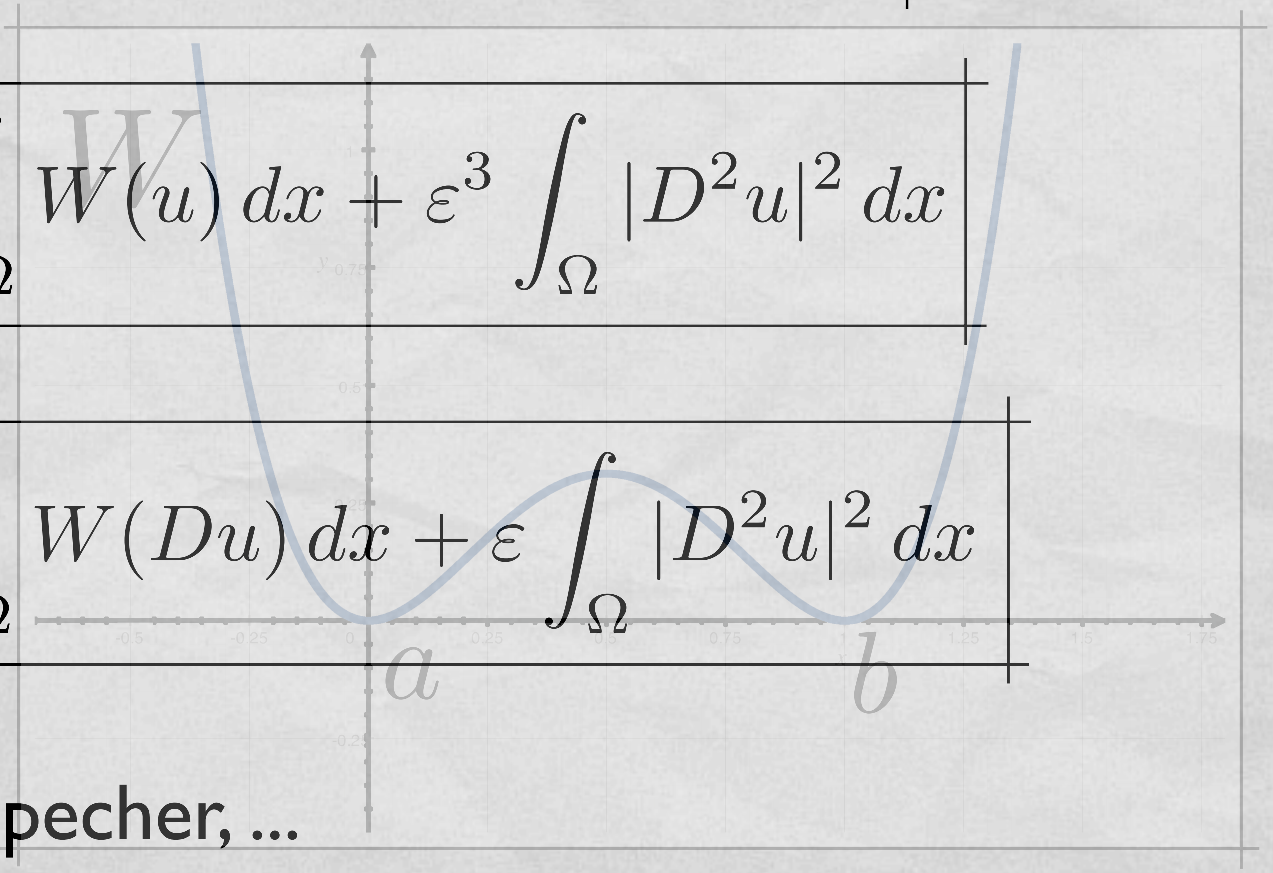
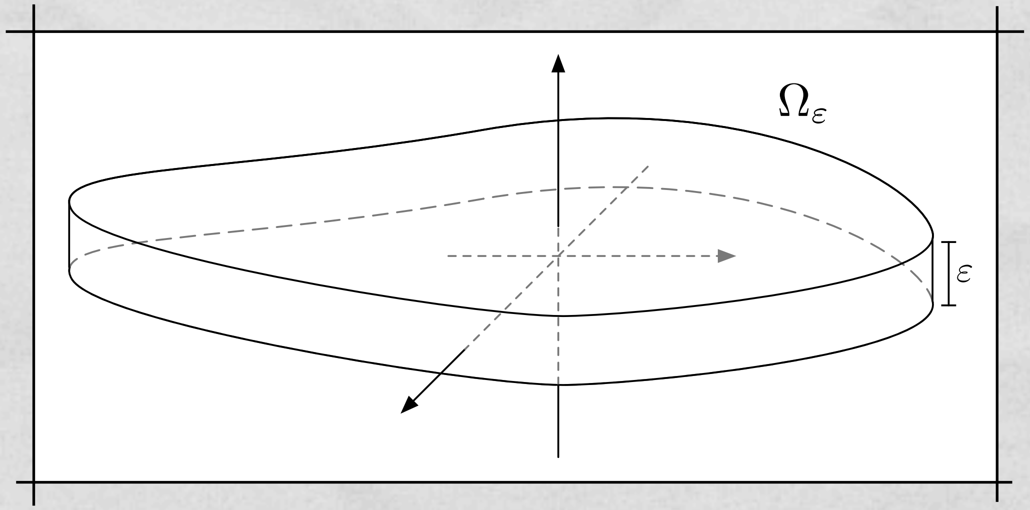


Alberti, Bouchitté, Garroni, Palatucci, Seppecher, ...

HISTORIC CONTEXT

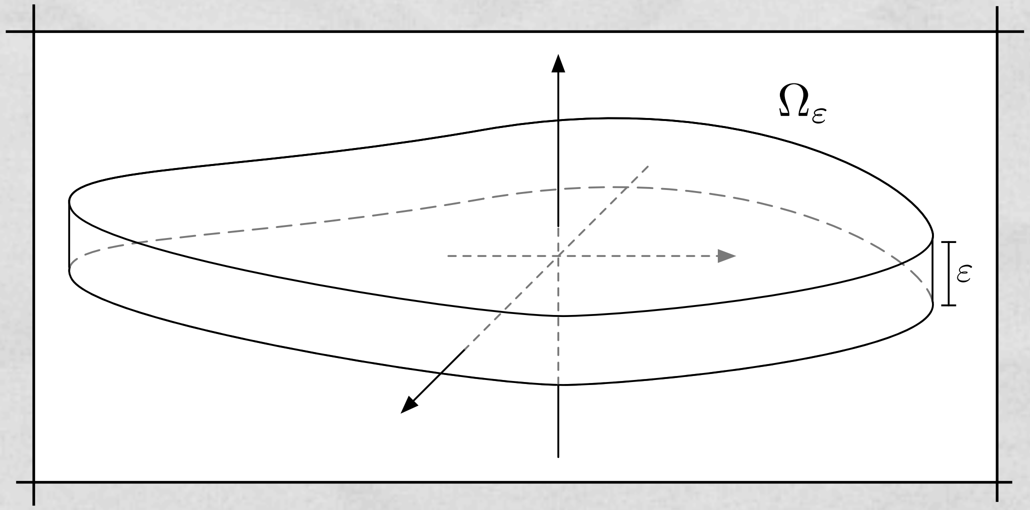
THIN FILMS

Instead of using the 3D Elasticity model on a very thin domain

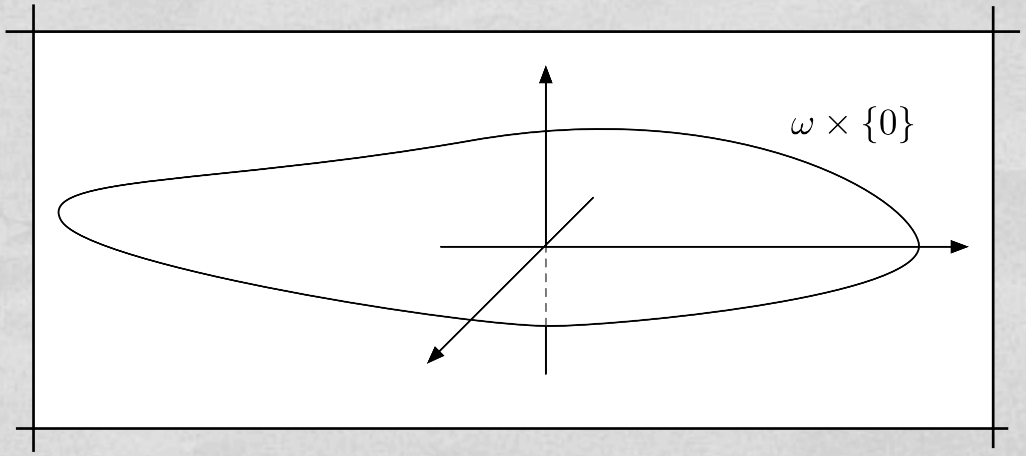


HISTORIC CONTEXT

Instead of using the 3D Elasticity model on a very thin domain



Deduce a 2D model from the 3D Elasticity Equations by letting the height of the domain go to 0



HISTORIC CONTEXT

THIN FILMS

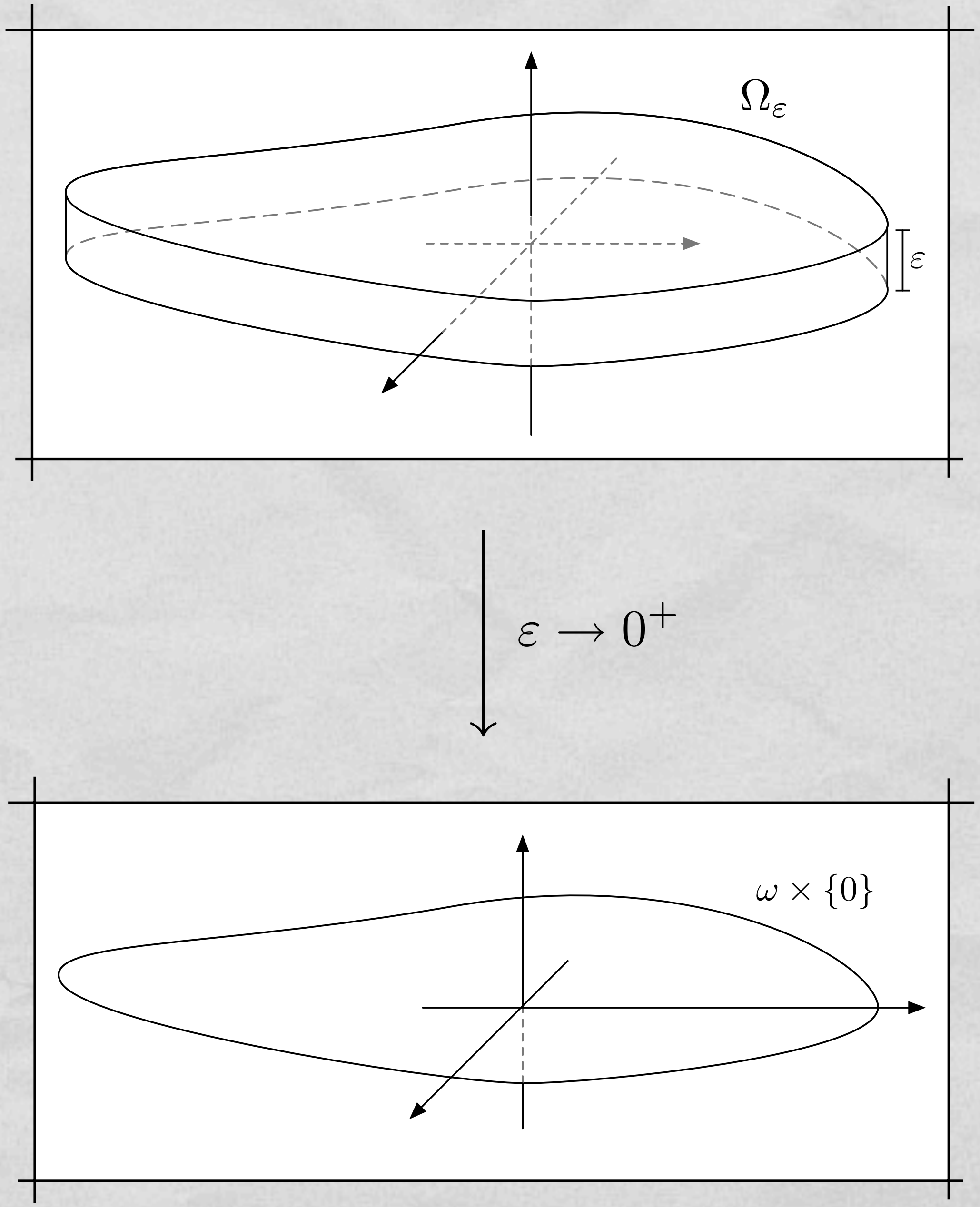
idea

3D Elasticity $\inf_{u \in H^1} \int_{\Omega_\varepsilon} W\left(\nabla u(\mathbf{x})\right) \mathbf{dx}$

infimizing sequence $\{u_\varepsilon\}$

$$u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u_0$$

Q: What is the limiting problem?
How to characterize it?



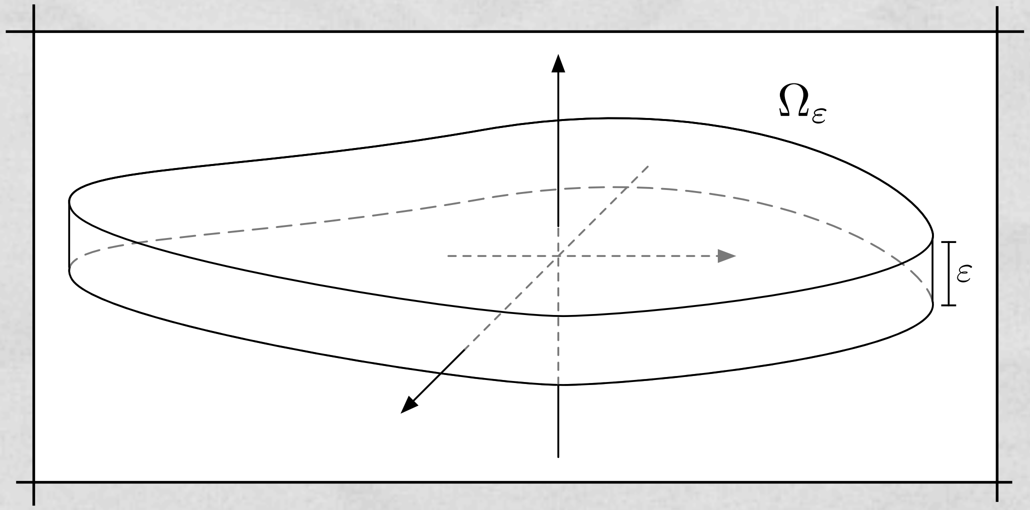
HISTORIC CONTEXT

THIN FILMS

Tool: Γ- limit

Step 1. Rescale

$$\boxed{\inf_{u_{eps} \in H^1} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W\left(\nabla u_\varepsilon(\mathbf{x})\right) \mathbf{dx}}$$



Historic Context

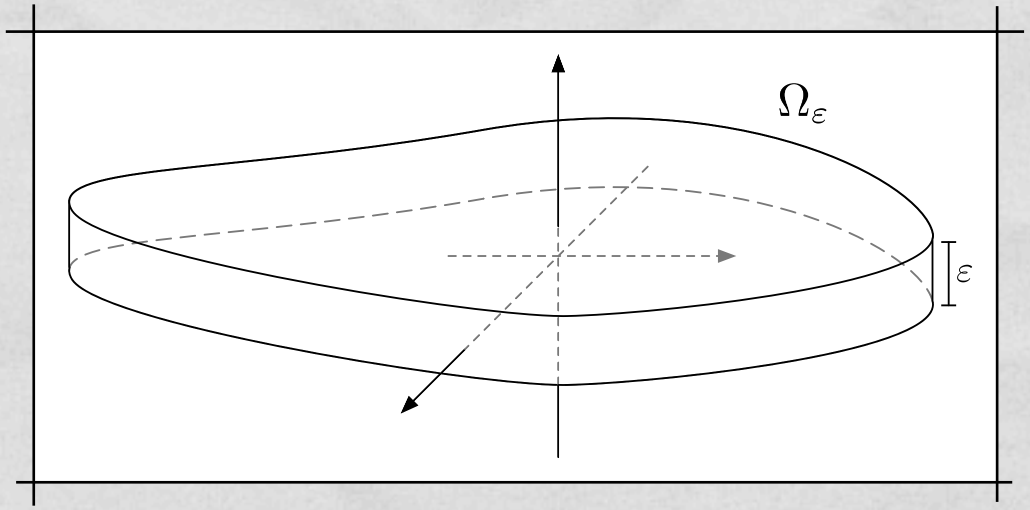
Tool: Γ- limit

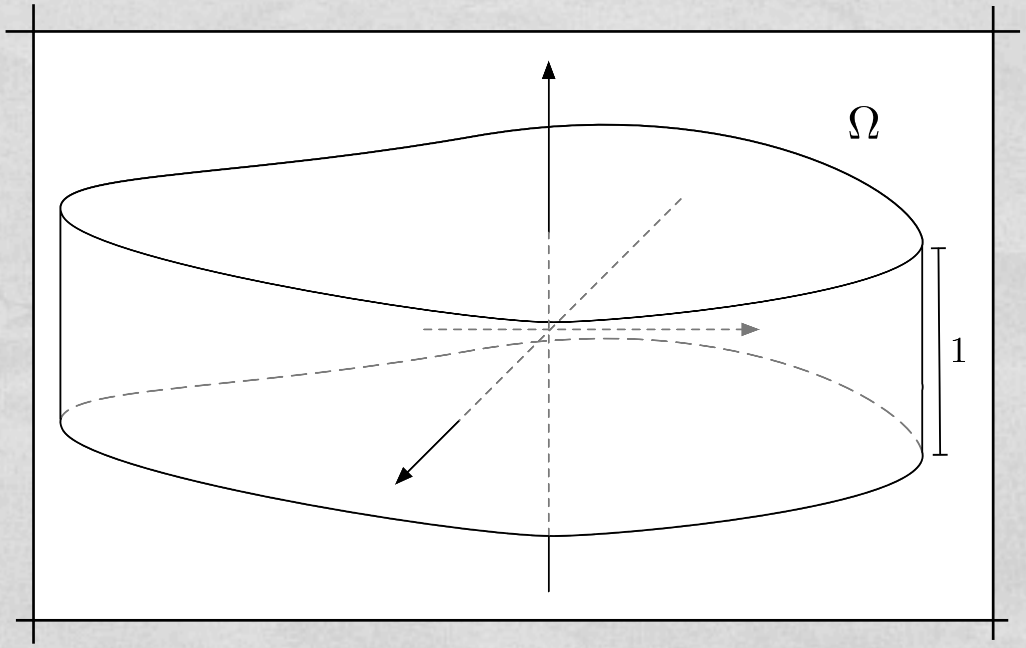
Step 1. Rescale

$$\boxed{\inf_{u_{eps}\in H^1}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W\left(\nabla u_\varepsilon(\mathbf{x})\right)\mathbf{dx}}$$

$$\downarrow$$

$$\boxed{\inf_{u_\varepsilon\in H^1}\int_{\Omega} W\left(\nabla' u_\varepsilon(x)|\tfrac{1}{\varepsilon}\partial_3 u_\varepsilon(x)\right)dx}$$





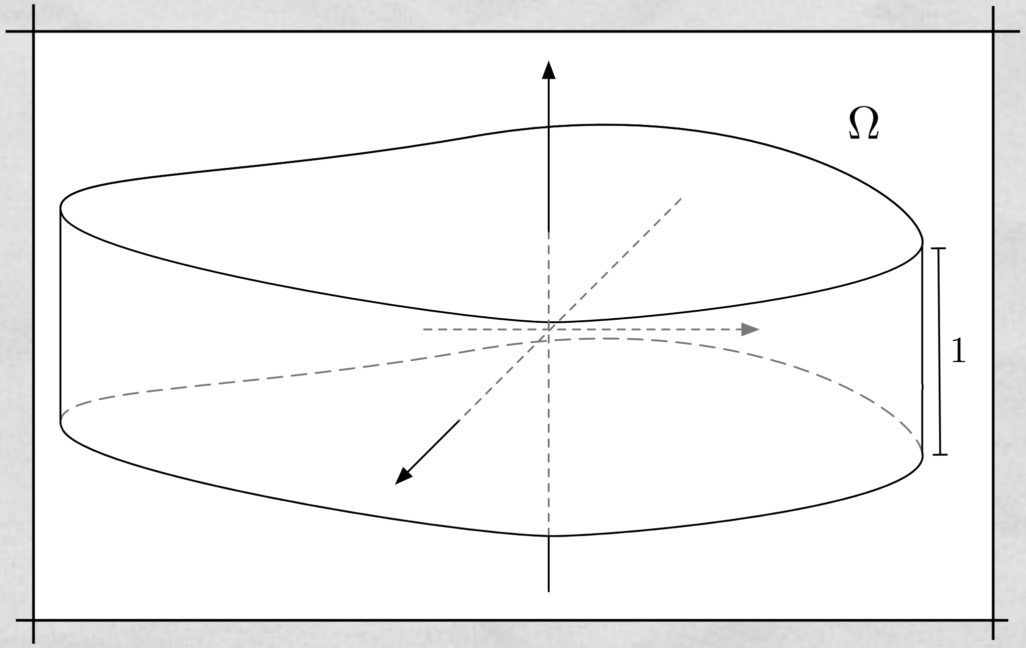
HISTORIC CONTEXT

THIN FILMS

Tool: Γ- limit

Step 2. Γ- limit

$$\inf_{u_\varepsilon \in H^1} \int_\Omega W\left(\nabla' u_\varepsilon(x) \middle| \tfrac{1}{\varepsilon}\partial_3 u_\varepsilon(x)\right) dx$$



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THIN FILMS

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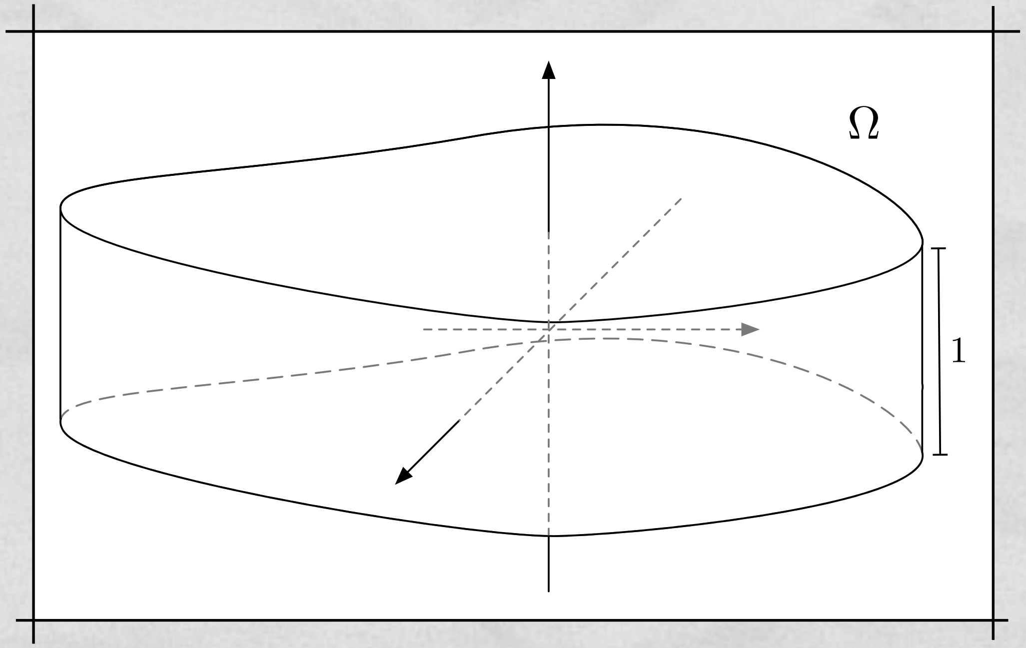
$$\boxed{\inf_{u_\varepsilon \in H^1} \int_\Omega W\left(\nabla' u_\varepsilon(x) | \tfrac{1}{\varepsilon}\partial_3 u_\varepsilon(x)\right) dx}$$

$$\Big\downarrow \Gamma\text{- limit}$$

$$\boxed{\min_{\substack{u \in H^1 \\ \partial_3 u = 0}} \int_\Omega QW_0\left(\nabla' u(x)\right) dx}$$



$$\boxed{W_0(\xi') := \min_{z \in \mathbb{R}^3} W(\xi' | z)}$$

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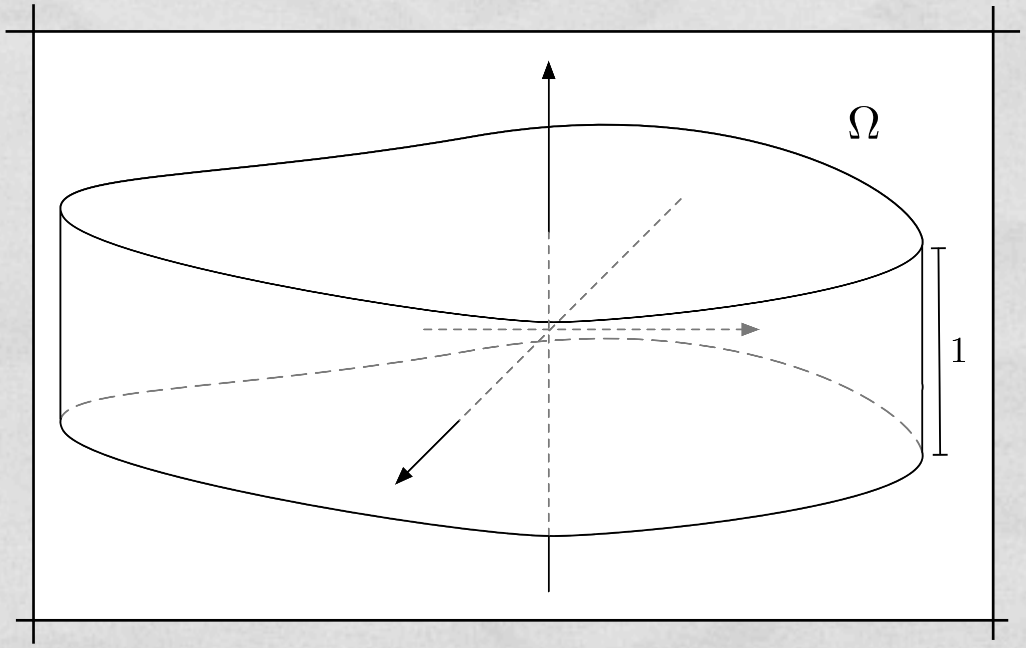

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HISTORIC CONTEXT

Ciarlet, Destuynder ‘78

Kohn, Vogelius ‘85

Acerbi, Buttazzo ‘86

Acerbi, Buttazzo, Percivale ‘88

Le Dret, Raoult ‘93

Bhattacharya, James ‘99

Braides, Bouchitté, Fonseca, Francfort, Friesecke, Leoni, Müller, ...

PHASE TRANSITIONS FOR THIN FILMS

Γ-LIMIT

$I_\varepsilon : X \longrightarrow [0,\infty]$, X metric space

$I_\varepsilon(u)$ Γ - converges to $I_0(u)$ in the topology of X

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(Γ- lim inf) $\forall u_\varepsilon \xrightarrow{X} u, \;\; I_0(u) \leqslant \liminf I_\varepsilon(u_\varepsilon)$

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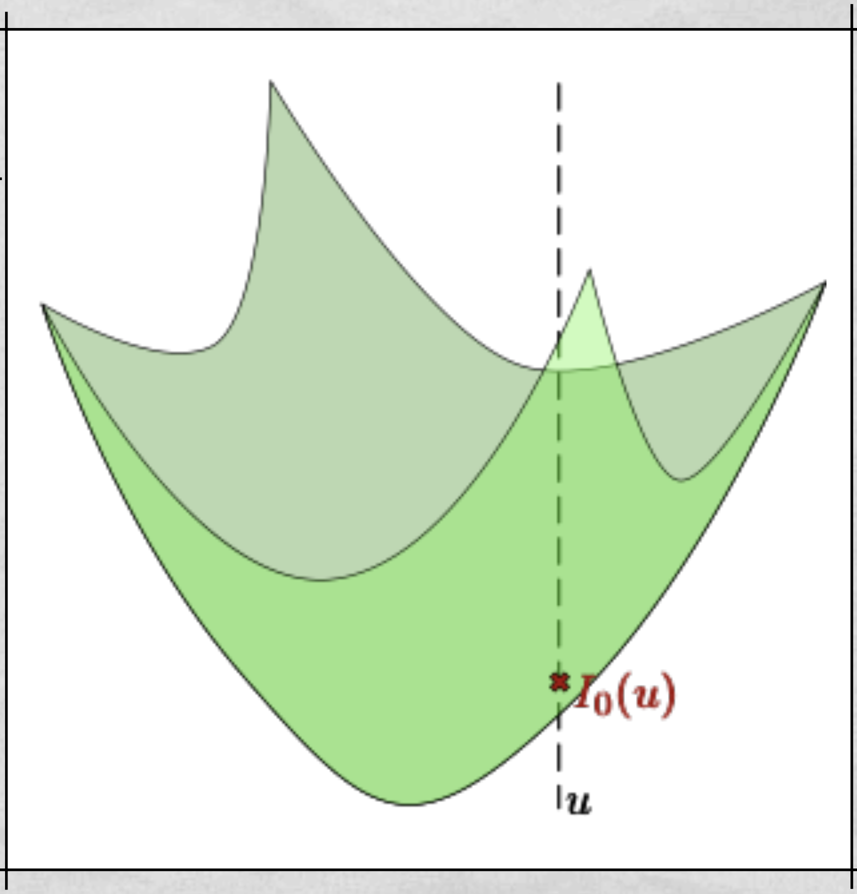

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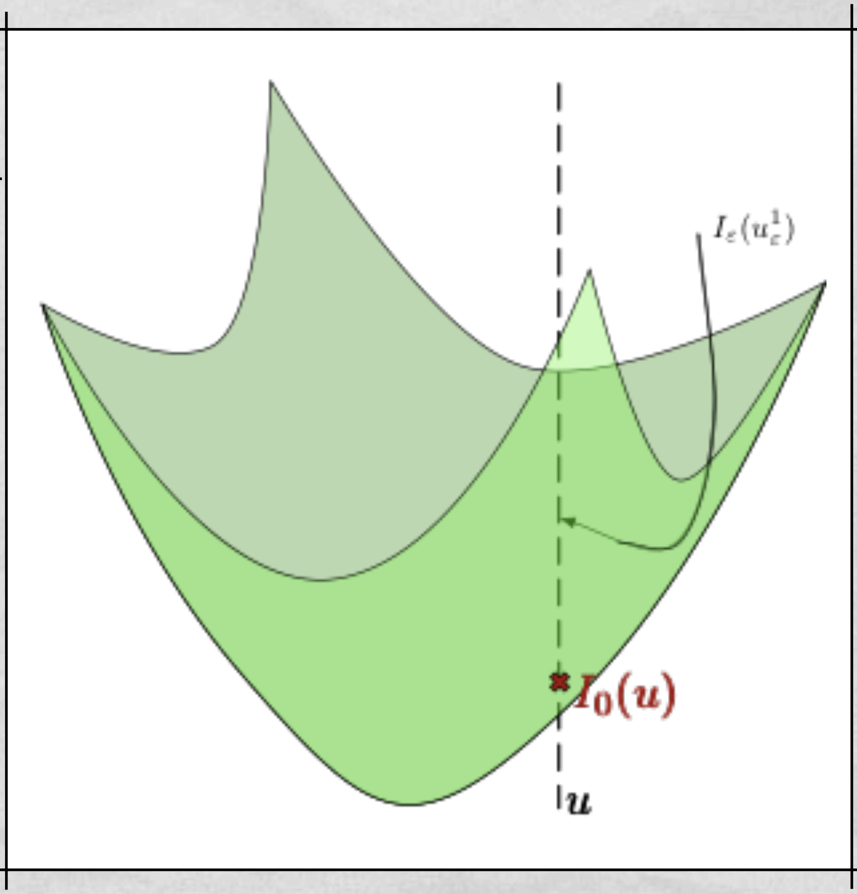

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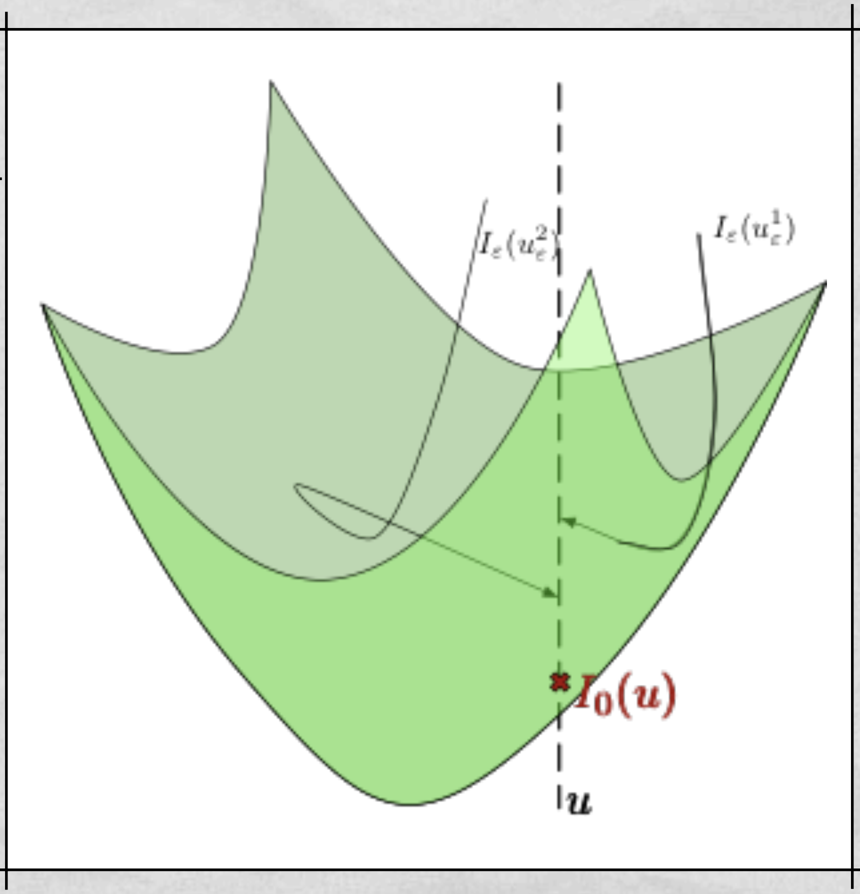

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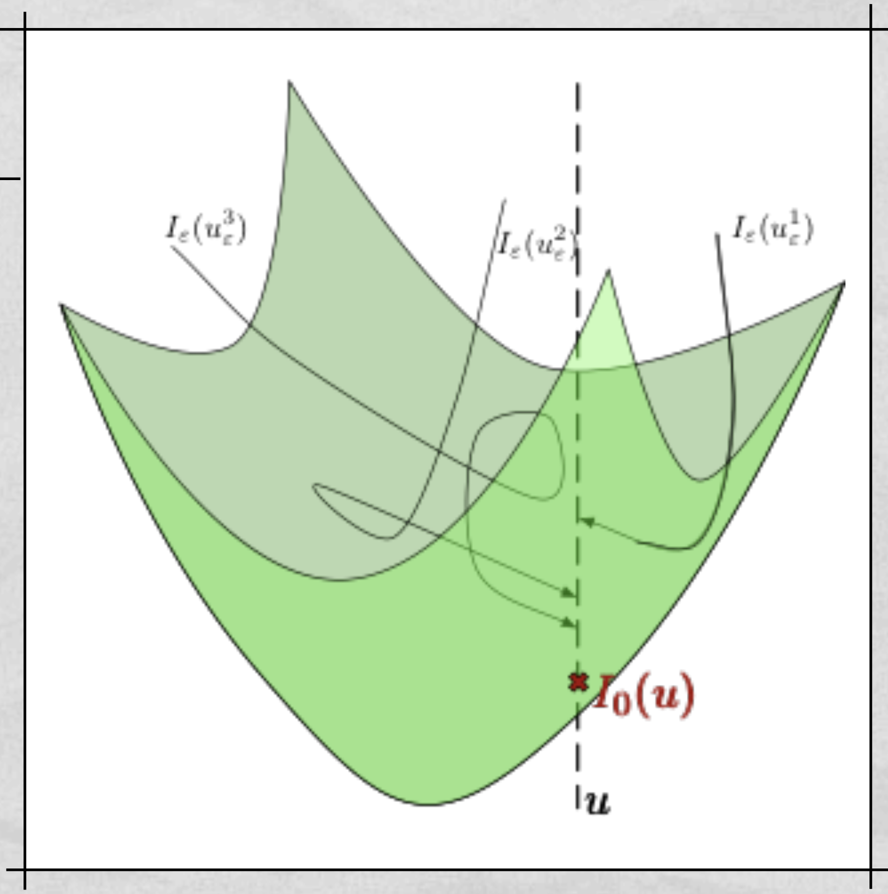

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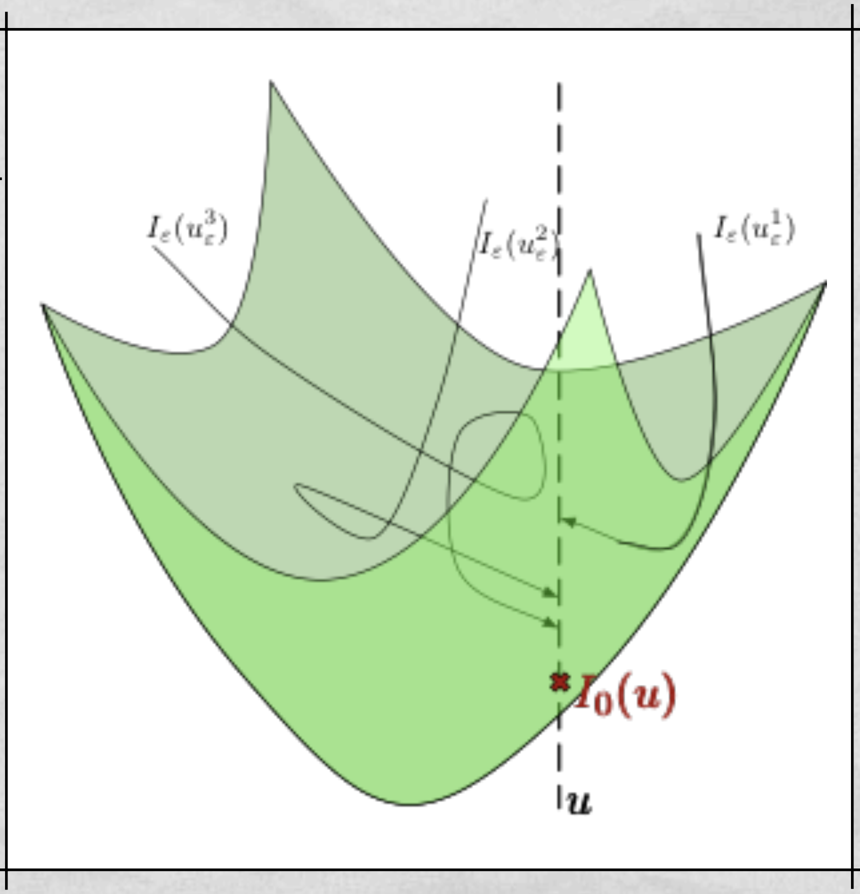
(Γ- lim sup) $\exists v_\varepsilon \xrightarrow{X} u,\ \ I_\varepsilon(v_\varepsilon) \longrightarrow I_0(u)$



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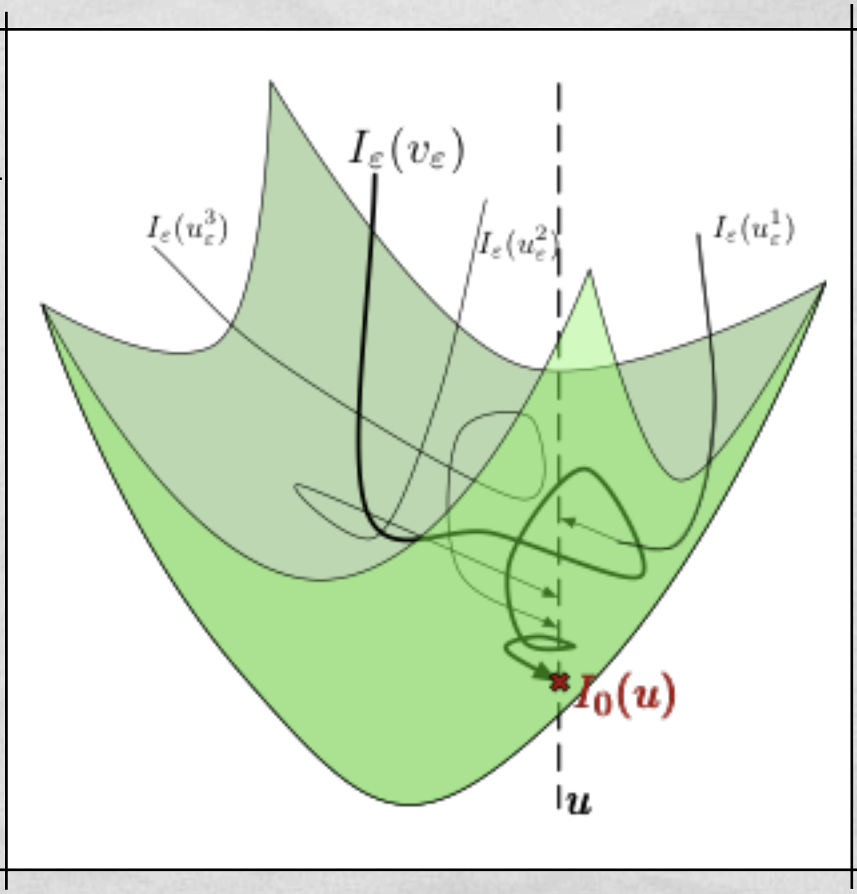
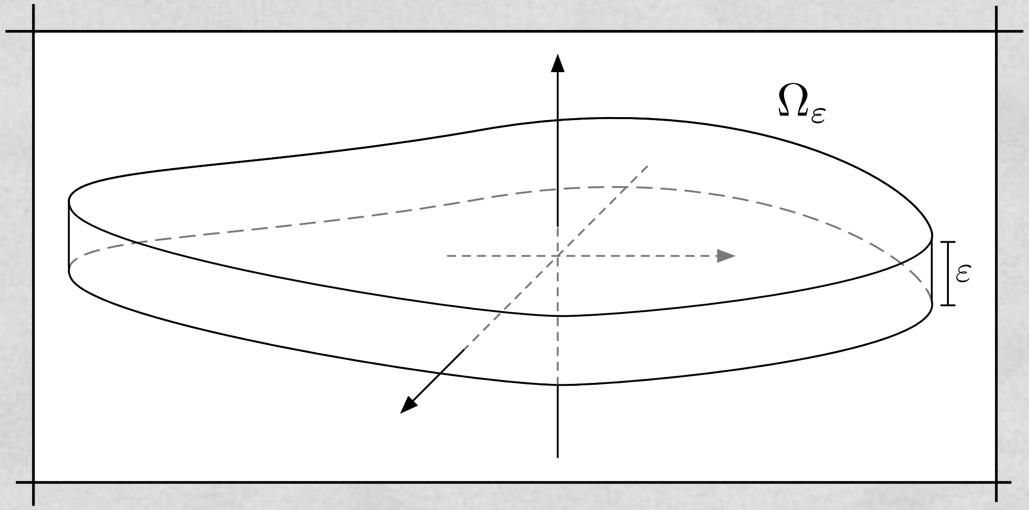
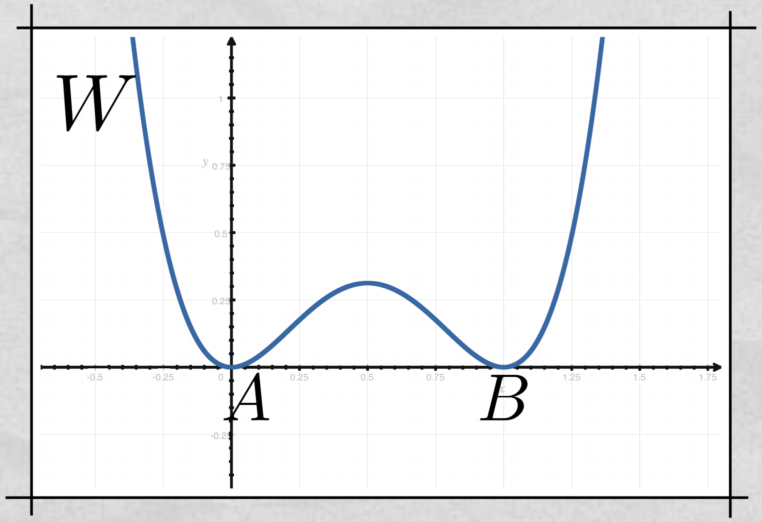


PHASE TRANSITIONS FOR THIN FILMS

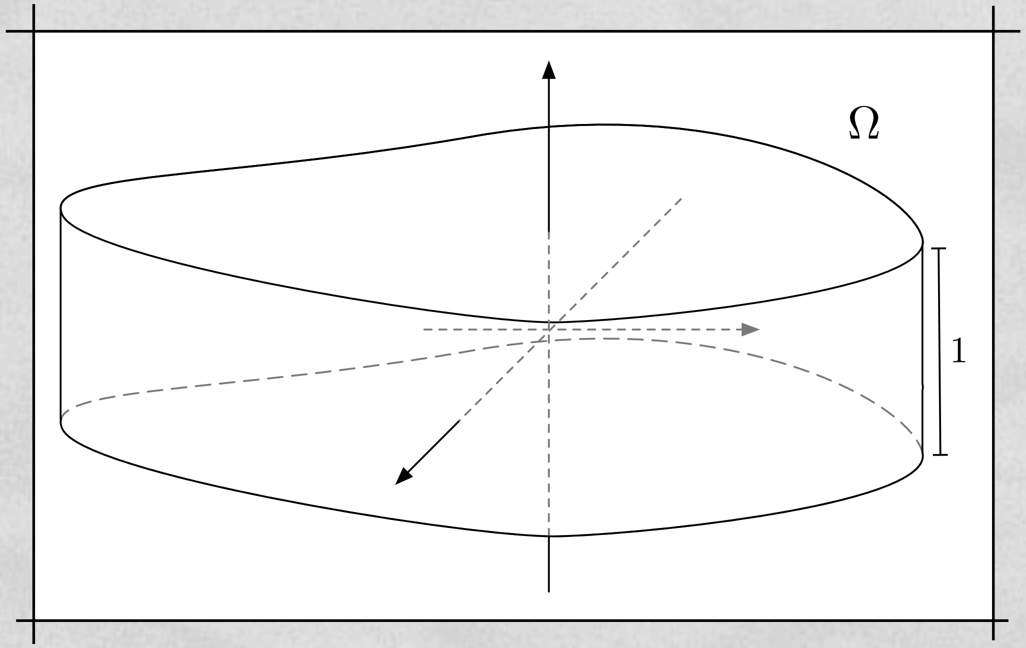
$$\mathbf{I}_\varepsilon^\gamma(\mathbf{u}) := \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon^\gamma}\int_{\Omega_\varepsilon} W\left(\nabla\mathbf{u}\right)\,d\mathbf{x} + \varepsilon^\gamma\int_{\Omega_\varepsilon}\left|\nabla^2\mathbf{u}\right|^2\,d\mathbf{x}\right)$$

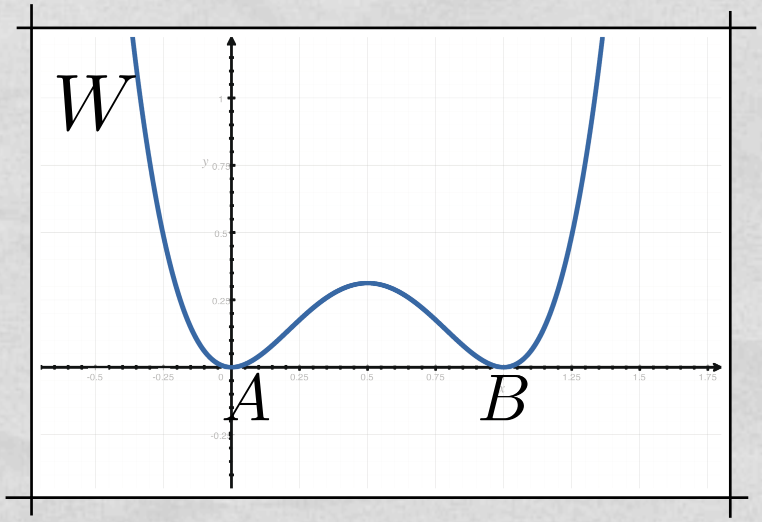




PHASE TRANSITIONS FOR THIN FILMS

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u | \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx$$

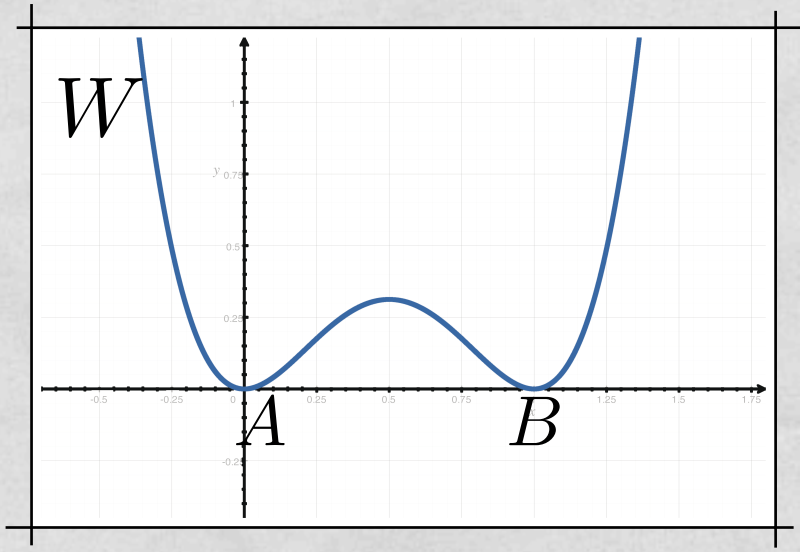




PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

$W : \mathbb{R}^{3\times 3} \longrightarrow [0,\infty)$ continuous double-well potential

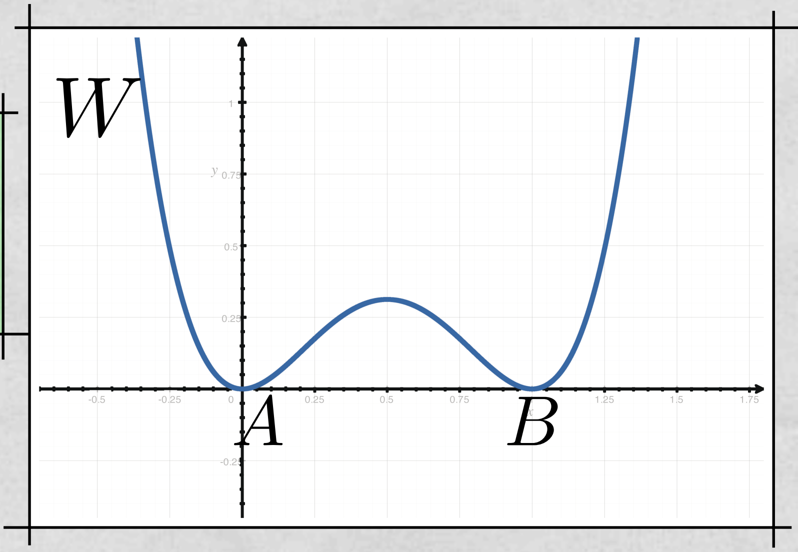


PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

$W : \mathbb{R}^{3\times 3} \longrightarrow [0, \infty)$ continuous double-well potential

$$(H_1) \qquad \frac{1}{C_1}\,|\xi|^p - C_1 \leqslant W(\xi) \leqslant C_1\,|\xi|^p + C_1, p \geqslant 2$$



PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

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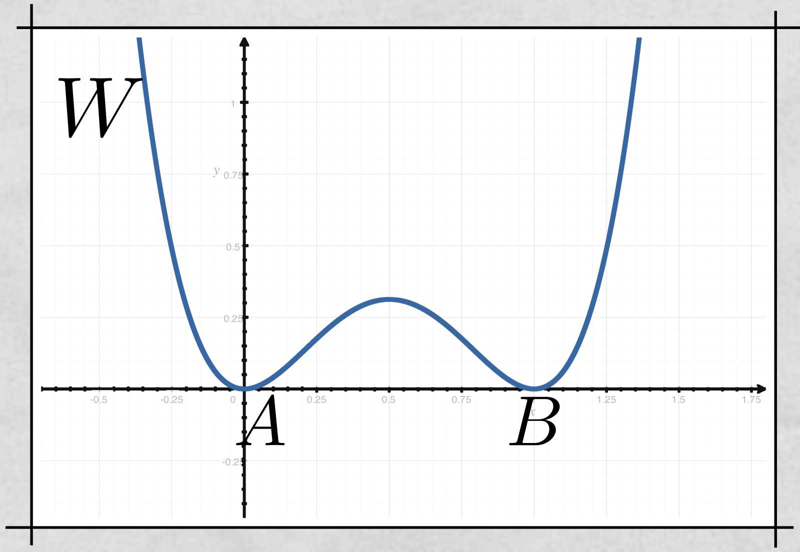
$$(H_1) \qquad \frac{1}{C_1}\,|\xi|^p - C_1 \leqslant W(\xi) \leqslant C_1\,|\xi|^p + C_1, p \geqslant 2$$

(H_2) $\quad W(\xi) = 0$ if and only if $\xi \in \{A, B\}$

$\quad A' - B' = a \otimes \nu$ rank-one connected



PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

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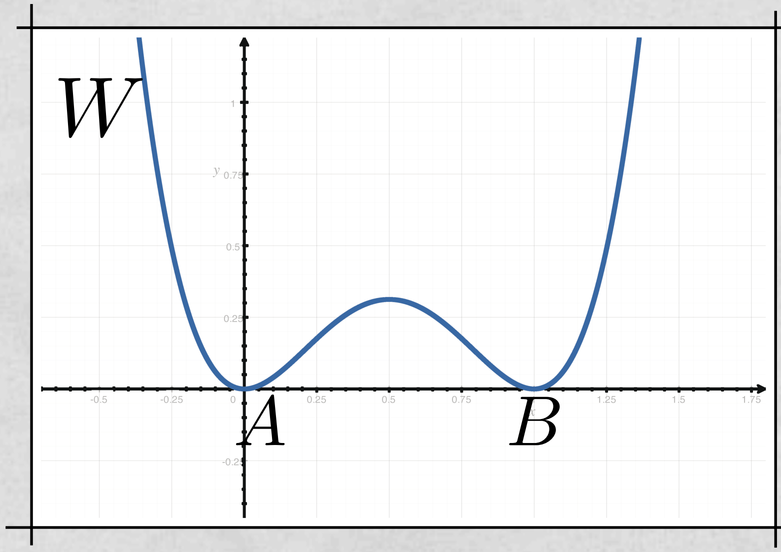

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$A' - B' = a \otimes \nu$ rank-one connected

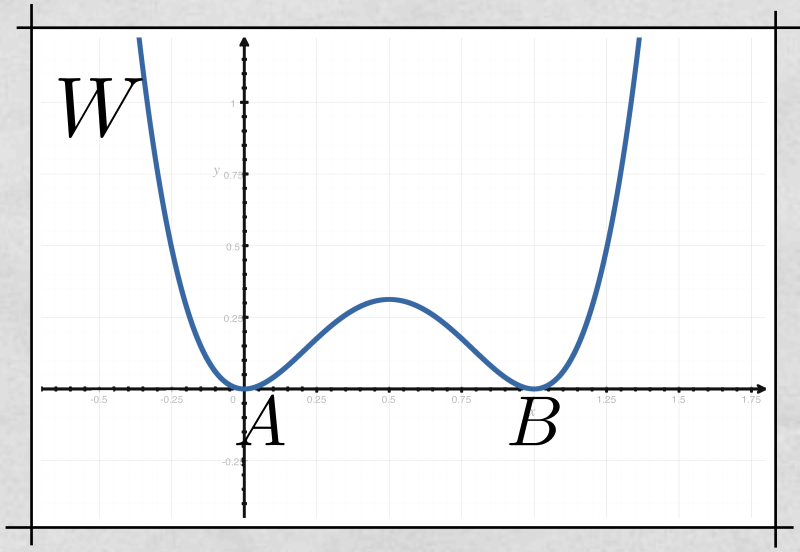
(H_3) $$\frac{1}{C_2}\operatorname{dist}\big(\xi,\{A,B\}\big)^p \leqslant W(\xi) \leqslant C_2 \operatorname{dist}\big(\xi,\{A,B\}\big)^p$$

if $\operatorname{dist}\big(\xi,\{A,B\}\big) \leqslant \rho$

PHASE TRANSITIONS IN THIN FILMS

Hypotheses on W

$W : \mathbb{R}^{3\times 3} \longrightarrow [0,\infty)$ continuous double-well potential



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(H_3) $\dfrac{1}{C_2}\operatorname{dist}\big(\xi, \{A,B\}\big)^p \leqslant W(\xi) \leqslant C_2 \operatorname{dist}\big(\xi, \{A,B\}\big)^p$

if $\operatorname{dist}\big(\xi, \{A,B\}\big) \leqslant \rho$

(H_4) best path from A' to B' does not depend on $\nu^\perp$ if $A' \neq B'$

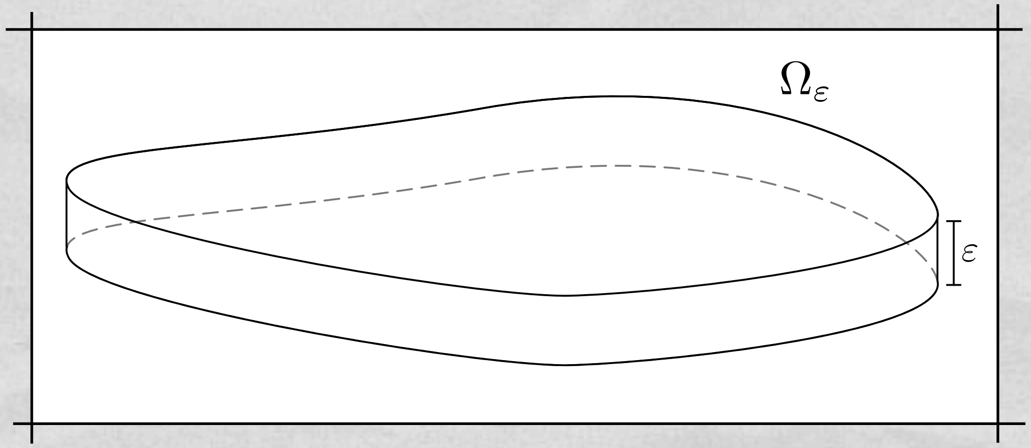
$W = W(|\xi'|, \xi_3)$ if $A' = B', A_3 \neq B_3$

PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx$$

ε **height of the domain**

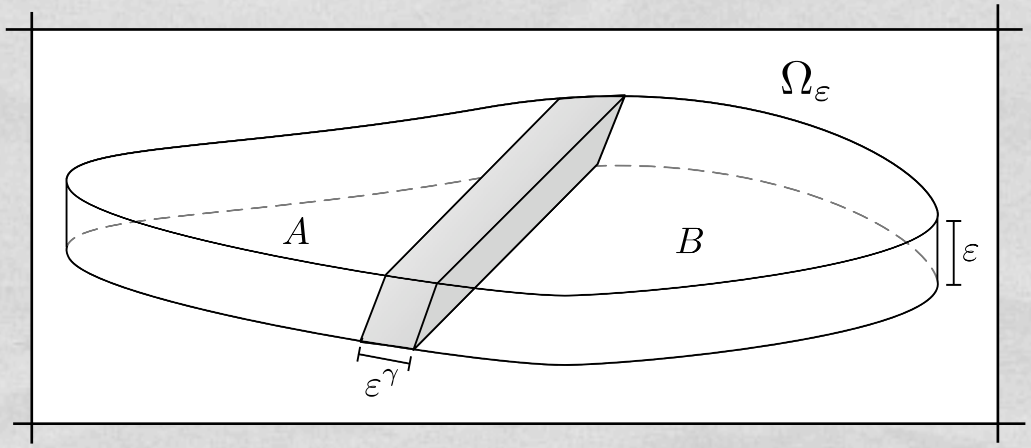


PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

$$\begin{aligned} I_\varepsilon^\gamma(u) := {} & \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ & + \varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}$$

ε	height of the domain
ε^γ	width of the transition layer



PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

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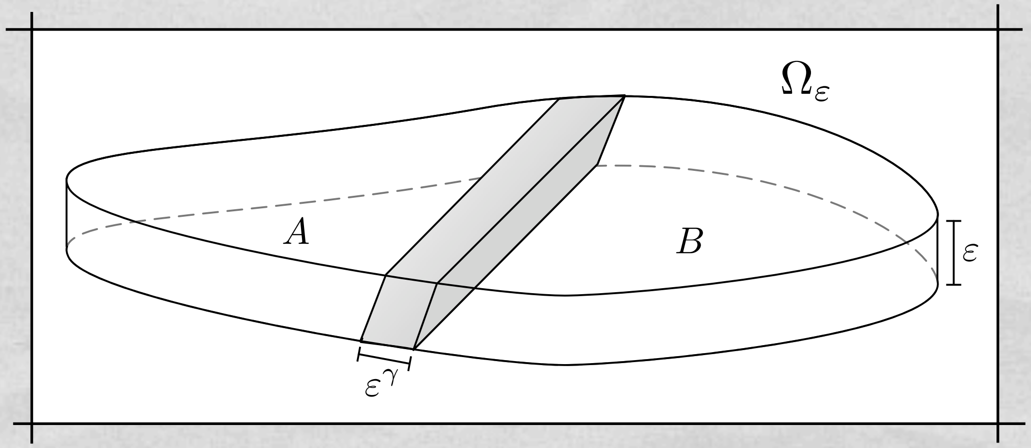
Three Regimes:

$\boxed{\gamma = 1}$ Critical Case



PHASE TRANSITIONS FOR THIN FILMS

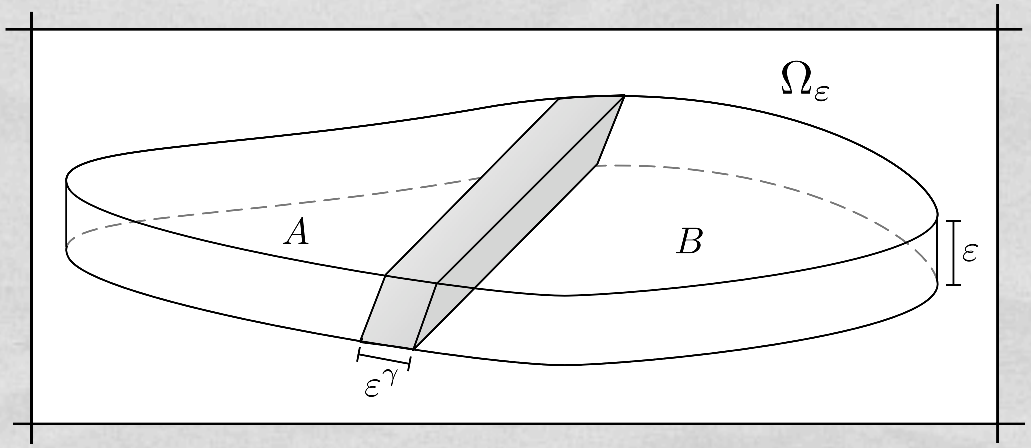
Identify Regimes

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right)dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right)dx \end{aligned}}$$

Three Regimes:

$\boxed{\gamma = 1}$ Critical Case

$\boxed{\gamma < 1}$ Subcritical Case



PHASE TRANSITIONS FOR THIN FILMS

Identify Regimes

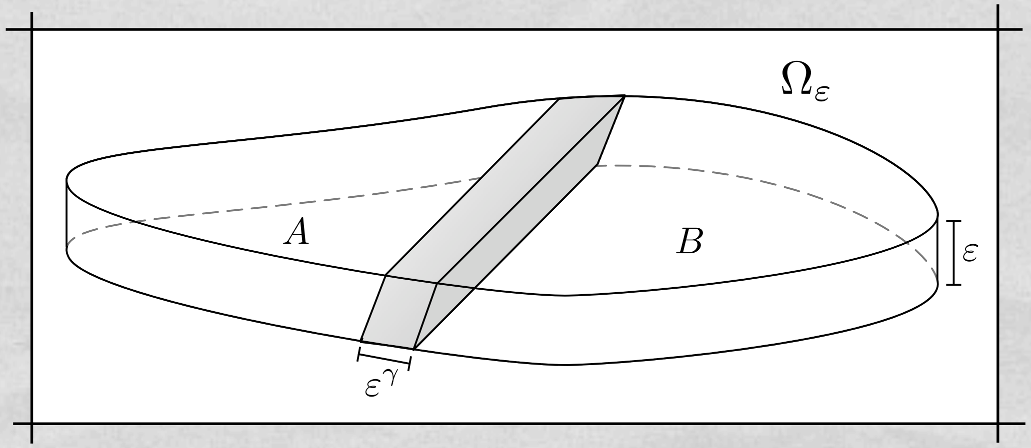
$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$

Three Regimes:

$\boxed{\gamma = 1}$ Critical Case

$\boxed{\gamma < 1}$ Subcritical Case

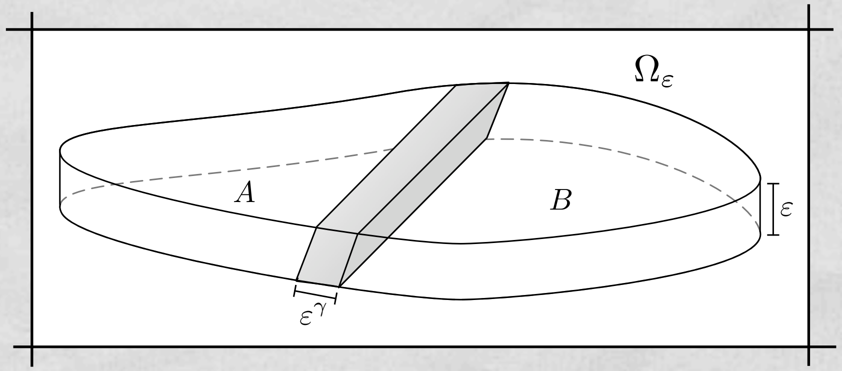
$\boxed{\gamma > 1}$ Supercritical Case



PHASE TRANSITIONS FOR THIN FILMS

Results

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right)dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right)dx$$

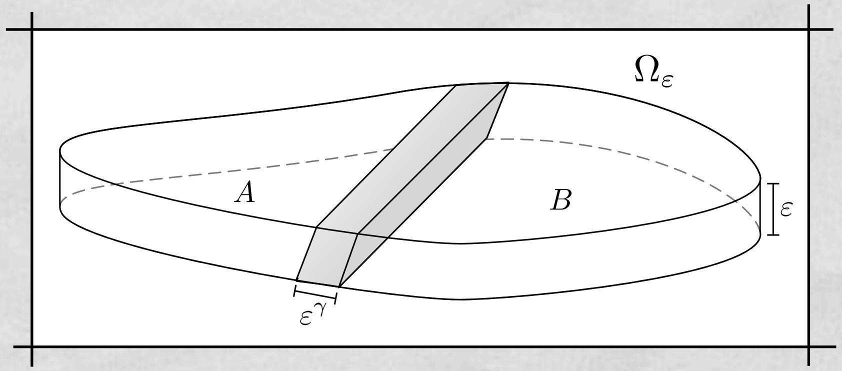


Results:

PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$



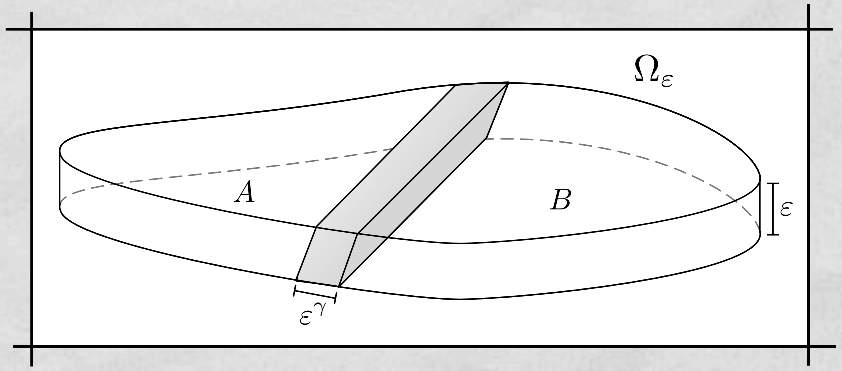
Results:

$\boxed{\gamma = 1} \longrightarrow \Gamma$ -limit

PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx \end{aligned}}$$

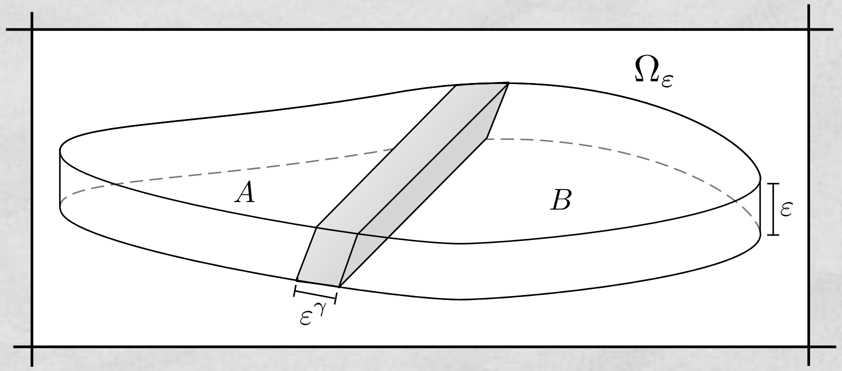


Results:

$\boxed{\gamma = 1}$ → Γ-limit → $A' \neq B'$

Phase Transitions for Thin Films

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$



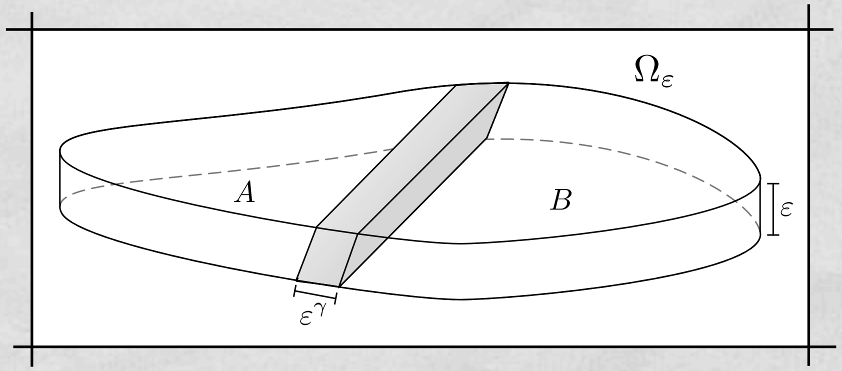
Results:

$$\begin{tikzcd} & & A' \neq B' \\ \boxed{\gamma = 1} \arrow[r] & \Gamma\text{-limit} \arrow[ur] \arrow[dr] & \\ & & A' = B', A_3 \neq B_3 \end{tikzcd}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx \end{aligned}}$$



Results:

$$\boxed{\gamma = 1} \longrightarrow \Gamma\text{-limit}$$

→ $A' \neq B'$

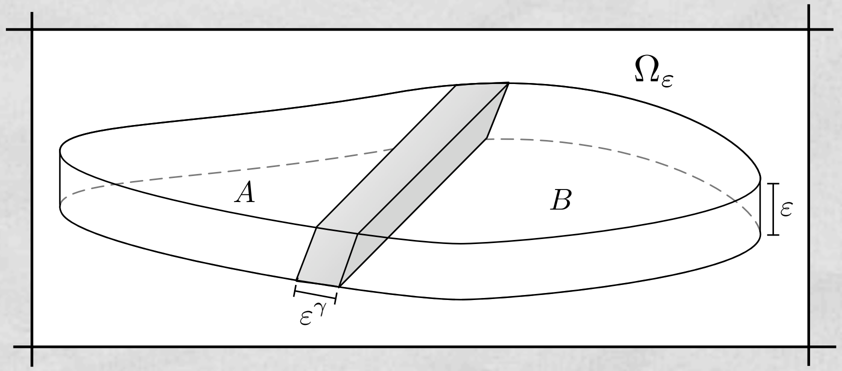
→ $A' = B', A_3 \neq B_3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\left(\nabla' u \,\middle|\, \tfrac{1}{\varepsilon}\partial_3 u\right)\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$



Results:

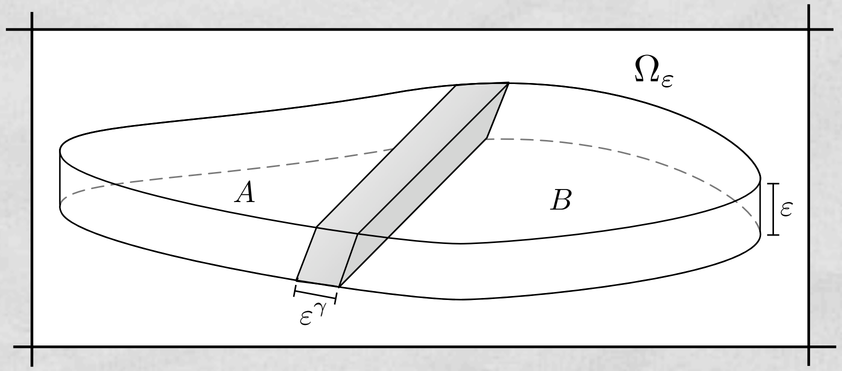
$$\begin{tikzcd} & & A' \neq B' \\ \boxed{\gamma = 1} \arrow[r] & \Gamma\text{-limit} \arrow[ur] \arrow[dr] & \\ & & A' = B', A_3 \neq B_3 \end{tikzcd}$$

$$A = \begin{array}{c} \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right) \\ \begin{array}{cc} \quad A' \quad & \quad\quad A_3 \end{array} \end{array}$$

Phase Transitions for Thin Films

Results

$$\boxed{\begin{aligned} I^{\gamma}_{\varepsilon}(u) := \frac{1}{\varepsilon^{\gamma}} \int_{\Omega} W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^{\gamma} \int_{\Omega} \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$



Results:

$\boxed{\gamma = 1} \longrightarrow$ Γ-limit

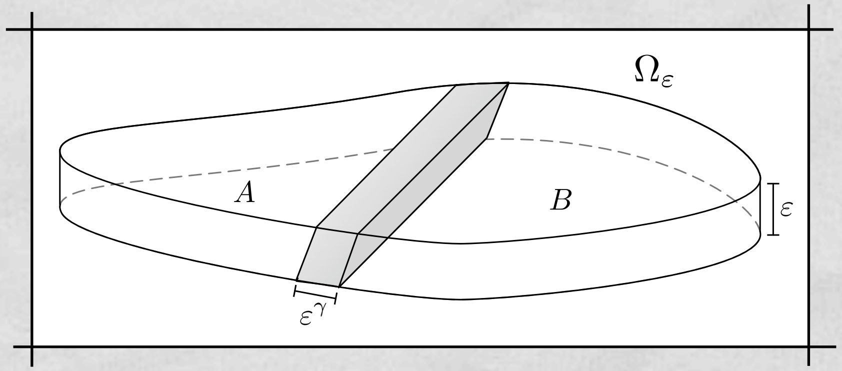
- $\nearrow$ $A' \neq B'$
- $\searrow$ $A' = B', A_3 \neq B_3$

$\boxed{\gamma < 1} \longrightarrow$ Γ-limit

PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$



Results:

$$\begin{tikzcd} & & A' \neq B' \\ \boxed{\gamma = 1} \arrow[r] & \Gamma\text{-limit} \arrow[ur] \arrow[dr] & \\ & & A' = B', A_3 \neq B_3 \end{tikzcd}$$

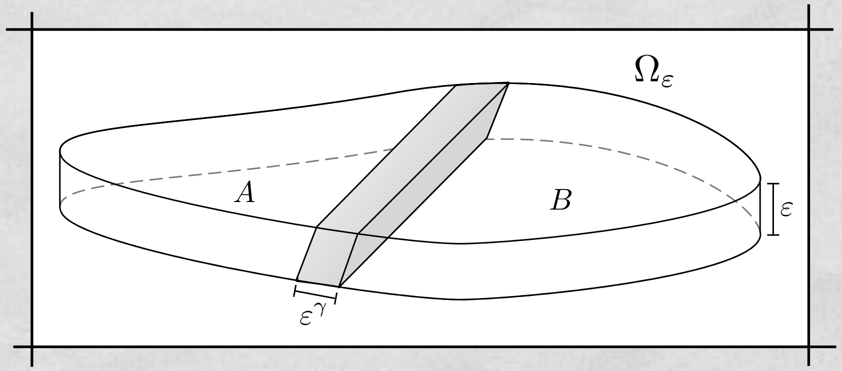
$$\begin{tikzcd} \boxed{\gamma < 1} \arrow[r] & \Gamma\text{-limit} \end{tikzcd}$$

$$\boxed{\gamma > 1}$$

PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right)dx + \varepsilon^\gamma\int_\Omega\left(|(\nabla')^2u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right)dx}$$



Results:

$\boxed{\gamma = 1}$ → Γ-limit

- → $A' \neq B'$
- → $A' = B', A_3 \neq B_3$

$\boxed{\gamma < 1}$ → Γ-limit

$\boxed{\gamma > 1}$ → $A - B = a \otimes \nu$ rank-one connected

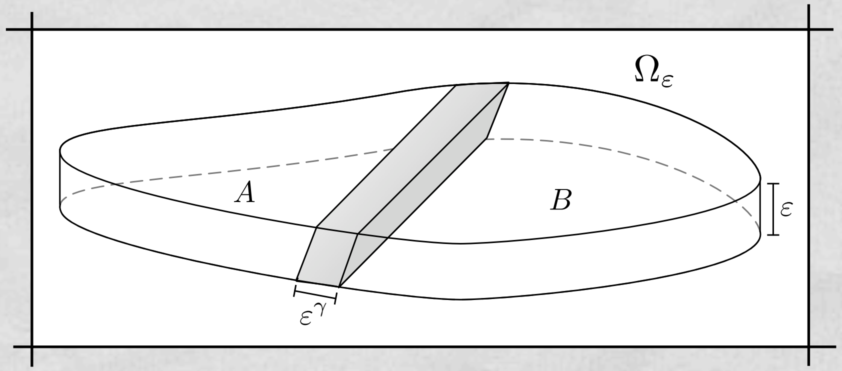
$A' \neq B'$

→ Γ-limit

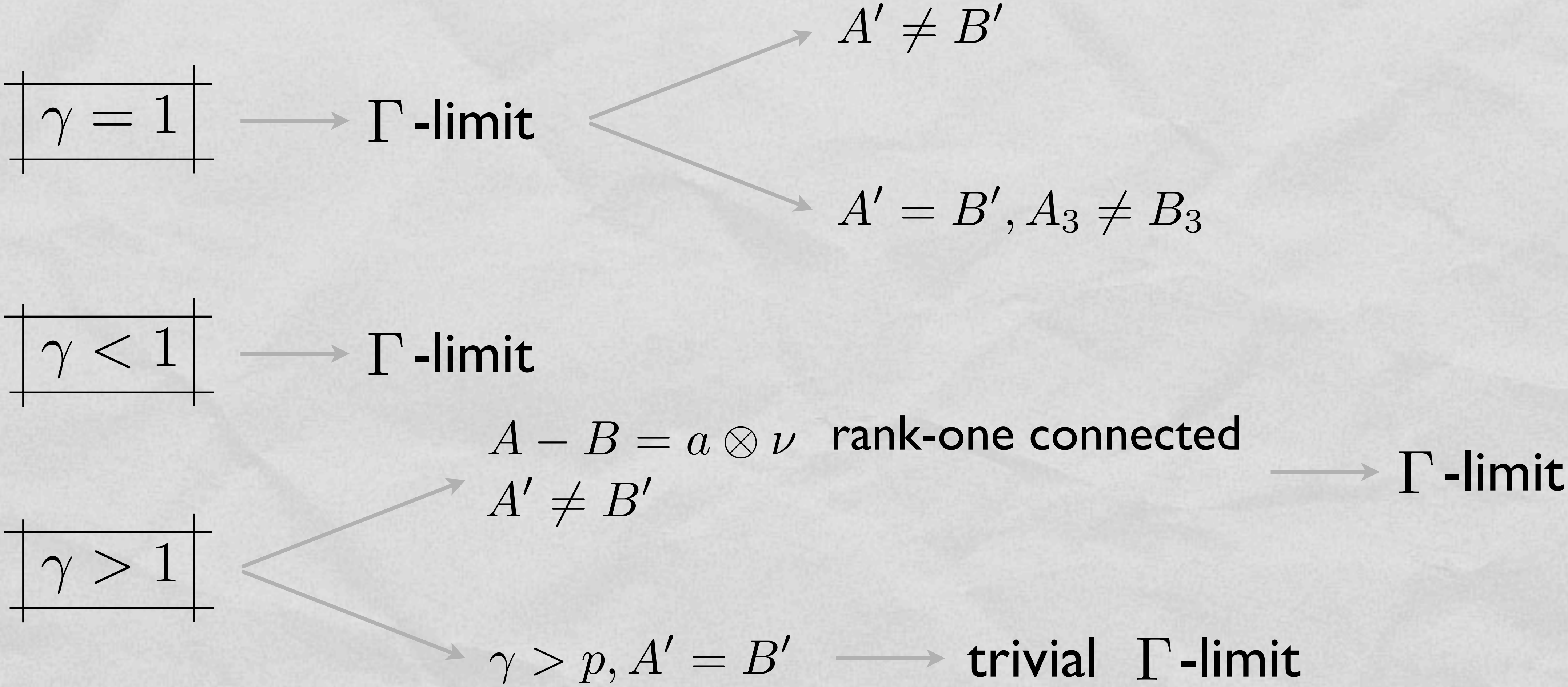
PHASE TRANSITIONS FOR THIN FILMS

Results

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx \end{aligned}}$$



Results:



$\boxed{\gamma = 1} \longrightarrow$ Γ-limit

$\longrightarrow A' \neq B'$

$\longrightarrow A' = B', A_3 \neq B_3$

$\boxed{\gamma < 1} \longrightarrow$ Γ-limit

$\boxed{\gamma > 1}$

$\longrightarrow A - B = a \otimes \nu$ rank-one connected
$A' \neq B'$ $\longrightarrow$ Γ-limit

$\longrightarrow \gamma > p, A' = B' \longrightarrow$ trivial Γ-limit

PHASE TRANSITIONS FOR THIN FILMS

Rigidity

Ball, James ‘87

$$\boxed{\begin{array}{l} u \in W^{1,\infty}(\Omega;\mathbb{R}^3) \\ \nabla u \in \{A, B\} \end{array}}$$

$\Longrightarrow$ $\boxed{A - B = a \otimes \nu \quad \text{rank-one connected}}$

PHASE TRANSITIONS FOR THIN FILMS

Rigidity

Ball, James '87

$$\boxed{\begin{array}{l} u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \nabla u \in \{A, B\} \end{array}}$$

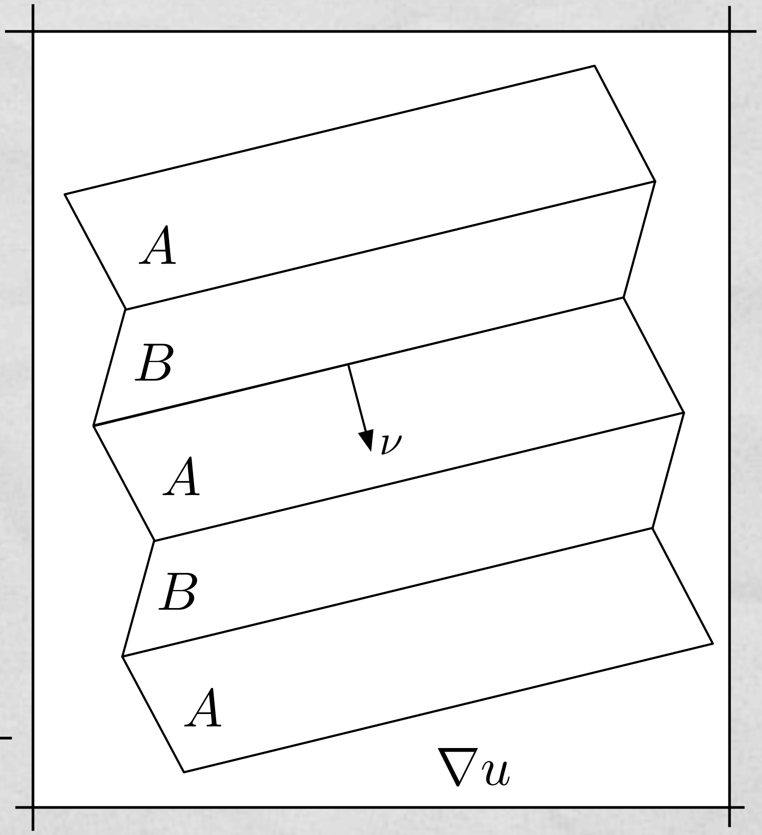
$$\Longrightarrow \boxed{\begin{array}{l} A - B = a \otimes \nu \quad \text{rank-one connected} \\ u(x) = \gamma_0 + Ax + h(x)a \\ \nabla h(x) = \chi_{\{\nabla u = B\}}(x)\nu \end{array}}$$

PHASE TRANSITIONS FOR THIN FILMS

Rigidity

Ball, James '87

$$u \in W^{1,\infty}(\Omega;\mathbb{R}^3)$$
$$\nabla u \in \{A, B\}$$



$\Longrightarrow$

$A - B = a \otimes \nu$ rank-one connected

$$u(x) = \gamma_0 + Ax + h(x)a$$

$$\nabla h(x) = \chi_{\{\nabla u = B\}}(x)\nu$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Rigidity

$$\boxed{\mathcal{V} := \left\{ (u, b) \in W^{1,\infty}(\Omega; \mathbb{R}^3) \times L^\infty(\Omega; \mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \quad (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}}$$

$$\boxed{\begin{aligned} u_\varepsilon &\to u \\ \tfrac{1}{\varepsilon}\partial_3 u_\varepsilon &\to b \end{aligned}}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Rigidity

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$$\boxed{\begin{array}{l} u_\varepsilon \to u \\ \frac{1}{\varepsilon}\partial_3 u_\varepsilon \to b \end{array}}$$

$$\left. (\nabla' u | b) \in BV(\Omega; \{A, B\}) \right\}$$

$$\Rightarrow \boxed{\begin{array}{c} u \in W^{1,\infty}(\Omega;\mathbb{R}^3) \\ \nabla u \in \{(A'|0),(B'|0)\} \end{array}}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

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$$\boxed{\begin{aligned} u_\varepsilon &\to u \\ \tfrac{1}{\varepsilon}\partial_3 u_\varepsilon &\to b \end{aligned}}$$

$$\Rightarrow \boxed{\begin{gathered} u \in W^{1,\infty}(\Omega; \mathbb{R}^3) \\ \nabla u \in \{(A'|0), (B'|0)\} \end{gathered}}$$

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}}_{A'} \underbrace{\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix}}_{A_3}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Rigidity

$$\mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \quad (\nabla' u | b) \in BV(\Omega;\{A,B\}) \Big\}$$

$u_\varepsilon \to u$

$\frac{1}{\varepsilon}\partial_3 u_\varepsilon \to b$

$$\Longrightarrow \quad \boxed{\begin{array}{c} u \in W^{1,\infty}(\Omega;\mathbb{R}^3) \\ \nabla u \in \big\{(A'|0),(B'|0)\big\} \end{array}}$$

Ball, James '87

$$\Longrightarrow \quad \boxed{A' - B' = a \otimes \nu \quad \text{rank-one connected}}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Rigidity

$$\mathcal{V} := \Big\{(u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0,$$

$$(\nabla' u | b) \in BV(\Omega;\{A,B\})\Big\}$$

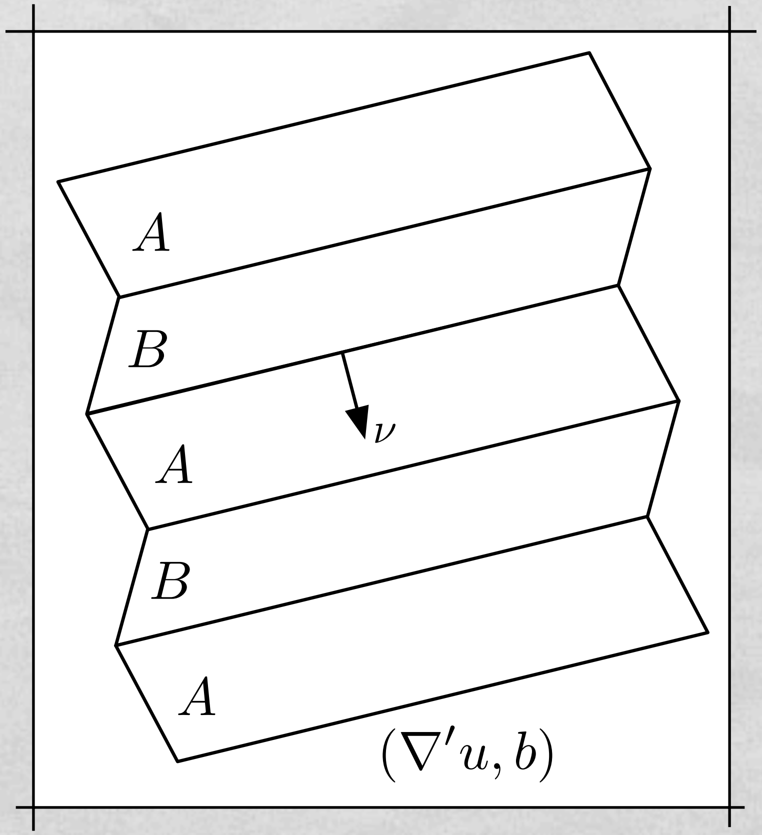
$$u_\varepsilon \to u$$

$$\tfrac{1}{\varepsilon}\partial_3 u_\varepsilon \to b$$

$$\Longrightarrow \quad \begin{array}{c} u \in W^{1,\infty}(\Omega;\mathbb{R}^3) \\ \nabla u \in \big\{(A'|0),(B'|0)\big\} \end{array}$$

Ball, James '87

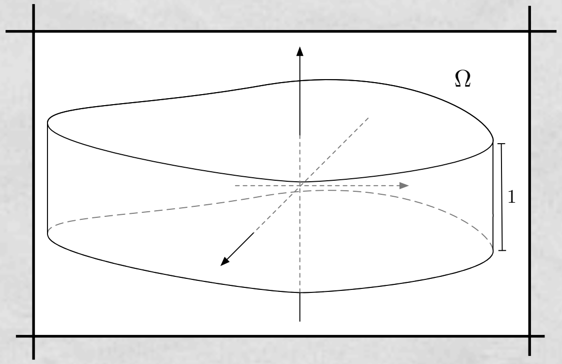
$$\Longrightarrow \quad A' - B' = a \otimes \nu \quad \text{rank-one connected}$$



PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

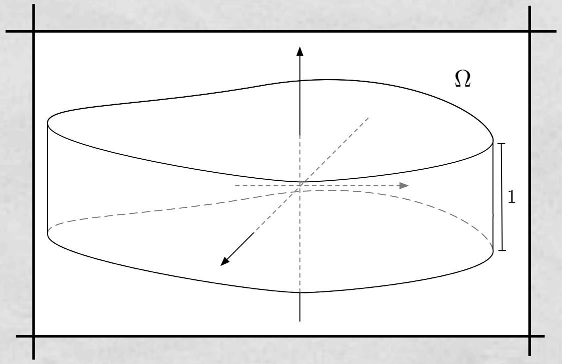
$$\boxed{\begin{aligned} I^1_\varepsilon(u) :=& \frac{1}{\varepsilon}\int_\Omega W\left(\nabla' u | \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ & + \varepsilon \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$



PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

$$I^1_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ + \varepsilon \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx$$



$$\mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \\ (\nabla' u | b) \in BV(\Omega;\{A,B\}) \Big\}$$

PHASE TRANSITIONS FOR THIN FILMS

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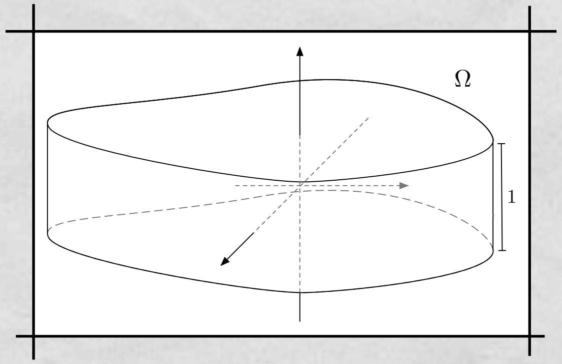

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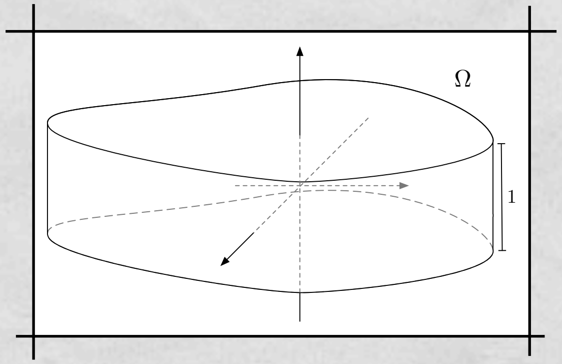
$u_\varepsilon \to u$

$\frac{1}{\varepsilon}\partial_3 u_\varepsilon \to b$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma = 1$ Critical Case

$$I^1_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega W\left(\nabla' u \middle| \frac{1}{\varepsilon}\partial_3 u\right) dx + \varepsilon\int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx$$



$$\Gamma - \lim_{\varepsilon \to 0+} I^1_\varepsilon(u,b) = \begin{cases} K^\star_\nu \operatorname{Per}_\omega\big(\{(\nabla' u|b) = A\}\big) & \text{if } (u,b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \; (\nabla' u|b) \in BV(\Omega;\{A,B\}) \Big\}$$

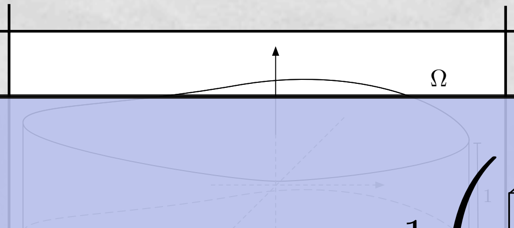
$u_\varepsilon \longrightarrow u$

$\frac{1}{\varepsilon}\partial_3 u_\varepsilon \longrightarrow b$

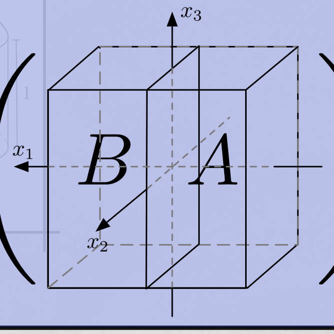
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

$$I^1_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega W\left(\nabla' u \Big| \frac{1}{\varepsilon}\partial_3 u\right) dx + \\ + \varepsilon \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx$$



$$K_\nu^\star := \Gamma - \liminf_{\varepsilon\to0+} I_\varepsilon^1 \left(\quad \right)$$



$$\Gamma - \lim_{\varepsilon\to0+} I^1_\varepsilon(u,b) = \begin{cases} K^\star_\nu \, \mathrm{Per}_\omega(\{(\nabla' u|b) = A\}) & \text{if } (u,b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

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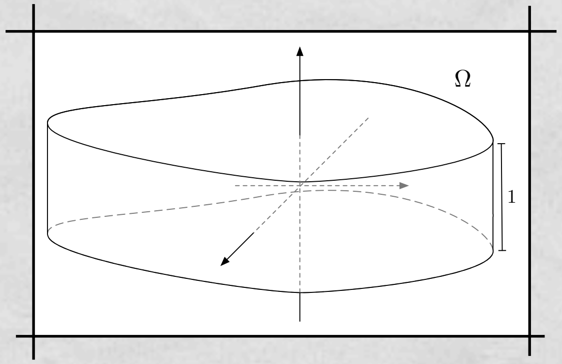
$u_\varepsilon \to u$

$\frac{1}{\varepsilon}\partial_3 u_\varepsilon \to b$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma < 1$ Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx$$



PHASE TRANSITIONS FOR THIN FILMS

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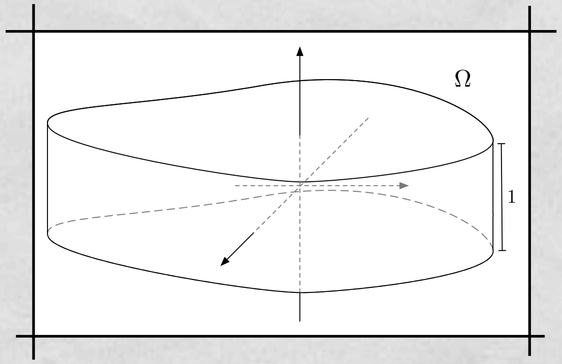
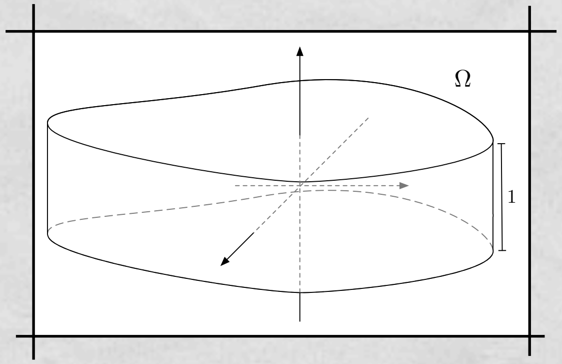


$$\mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \\ (\nabla' u | b) \in BV(\Omega;\{A,B\}) \Big\}$$

Phase Transitions for Thin Films

$\gamma < 1$ Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx$$



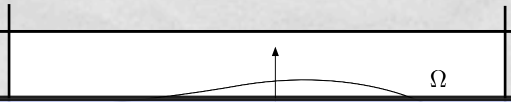
$$\Gamma - \lim_{\varepsilon \to 0+} I_\varepsilon^\gamma(u,b) = \begin{cases} K_{2D}\,\mathrm{Per}_\omega\big(\{(\nabla' u|b) = A\big) & \text{if } (u,b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \\ (\nabla' u|b) \in BV(\Omega;\{A,B\}) \Big\}$$

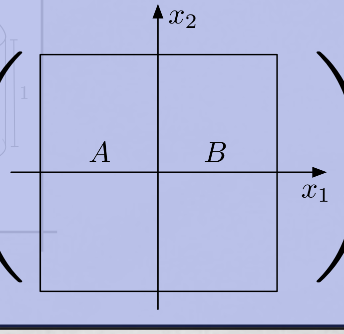
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma < 1}$ Subcritical Case

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right)dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right)$$



$$K_{2D} := \Gamma - \liminf_{\varepsilon \to 0+} I_\varepsilon^\gamma \left(\quad \right)$$



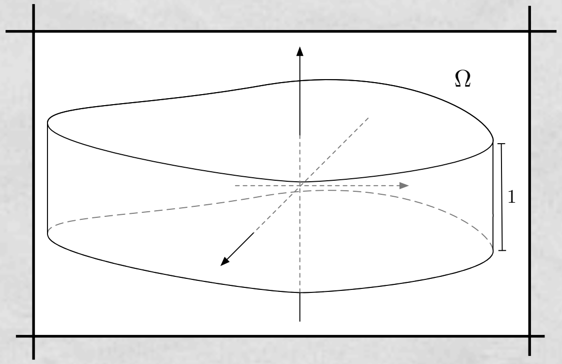
$$\Gamma - \lim_{\varepsilon \to 0+} I_\varepsilon^\gamma(u,b) = \begin{cases} \boldsymbol{K_{2D}}\,\mathrm{Per}_\omega(\{(\nabla' u|b) = A) & \text{if } (u,b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \left\{(u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \right. \\ \left. (\nabla' u|b) \in BV(\Omega;\{A,B\})\right\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma > 1}$ Supercritical Case $A - B = a \otimes \nu$ rank-1 connected, $A' \neq B'$

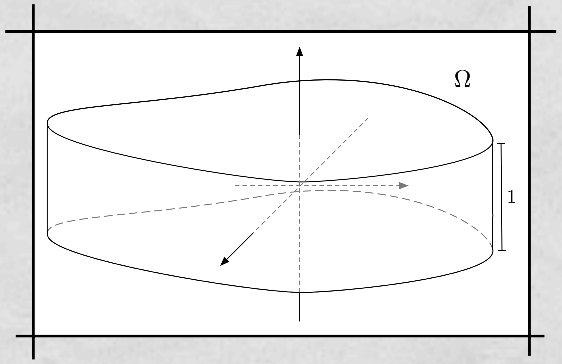
$$\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u | \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx \end{aligned}$$



PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma > 1}$ **Supercritical Case** $A - B = a \otimes \nu$ rank-1 connected, $A' \neq B'$

$$\boxed{\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma} \int_\Omega W\left(\nabla' u \middle| \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx \end{aligned}}$$

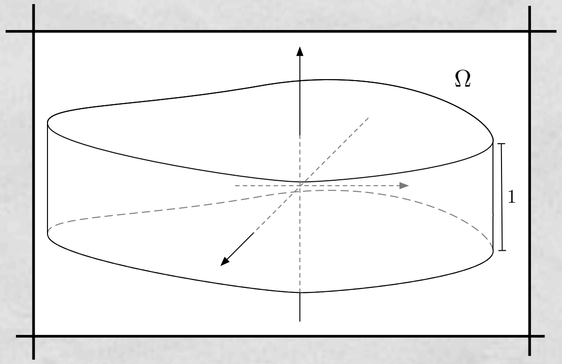


$$\boxed{\begin{aligned} \mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \\ (\nabla' u | b) \in BV(\Omega; \{A, B\}) \Big\} \end{aligned}}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma > 1}$ Supercritical Case $A - B = a \otimes \nu$ rank-1 connected, $A' \neq B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right) dx + \varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx$$



$$\Gamma - \lim_{\varepsilon \to 0+} I_\varepsilon^\gamma(u,b) = \begin{cases} K^\gamma \,\mathrm{Per}_\omega\big(\{(\nabla' u|b) = A\big) & \text{if } (u,b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\mathcal{V} := \Big\{ (u,b) \in W^{1,\infty}(\Omega;\mathbb{R}^3) \times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u = \partial_3 b = 0, \\ (\nabla' u|b) \in BV(\Omega;\{A,B\}) \Big\}$$

PHASE TRANSITIONS FOR THIN FILMS

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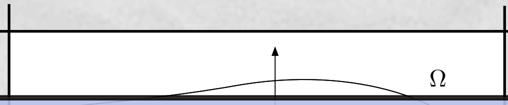
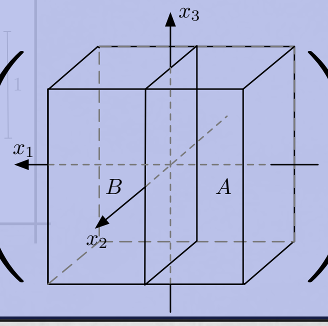


$$K^\gamma := \Gamma - \liminf_{\varepsilon \to 0+} I_\varepsilon^\gamma \left(\quad \right)$$



$$\Gamma - \lim_{\varepsilon \to 0+} I_\varepsilon^\gamma(u,b) = \begin{cases} \boldsymbol{K^\gamma}\, \mathrm{Per}_\omega(\{(\nabla' u|b) = A) & \text{if } (u,b) \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

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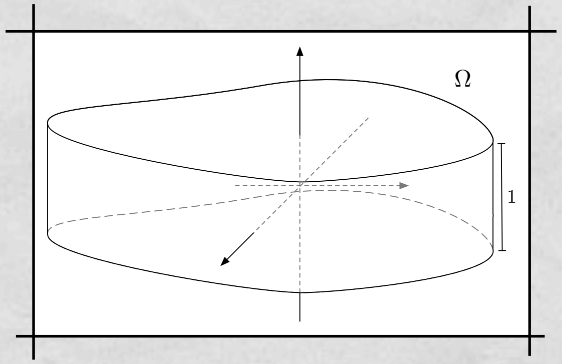
PHASE TRANSITIONS FOR THIN FILMS

$\gamma > 1$ Supercritical Case

$\gamma > p, A' = B'$

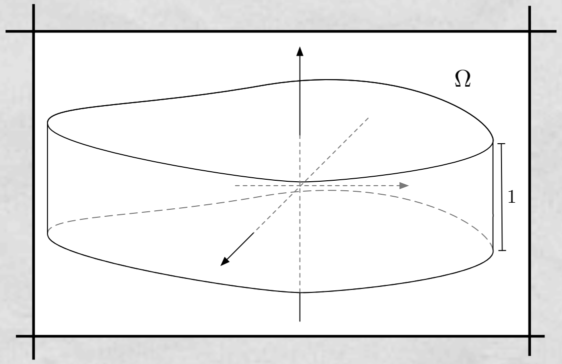
$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right)dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2 \right) dx$$



Phase Transitions for Thin Films

$\gamma > 1$ Supercritical Case $\gamma > p, A' = B'$

$$\begin{aligned} I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u | \tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ +\varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx \end{aligned}$$

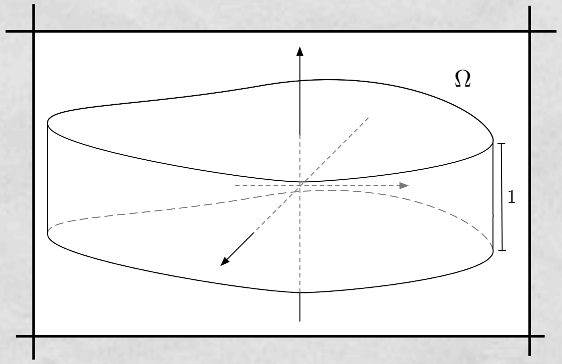


$$\overline{\mathcal{V}} := \{(u,b) \ : \ \nabla u = (A',0), b \in \{A_3, B_3\}\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\gamma > 1$ Supercritical Case $\gamma > p, A' = B'$

$$I_\varepsilon^\gamma(u) := \frac{1}{\varepsilon^\gamma}\int_\Omega W\left(\nabla' u|\tfrac{1}{\varepsilon}\partial_3 u\right) dx + \\ + \varepsilon^\gamma \int_\Omega \left(|(\nabla')^2 u|^2 + \left|\frac{1}{\varepsilon}\nabla'(\partial_3 u)\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{1}{\varepsilon}\partial_3^2 u\right|^2\right) dx$$



$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon^\gamma(u,b) = \begin{cases} 0 & \text{if } (u,b) \in \overline{\mathcal{V}}, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\overline{\mathcal{V}} := \{(u,b) \,:\, \nabla u = (A', 0), b \in \{A_3, B_3\}\}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 1. Γ- lim inf

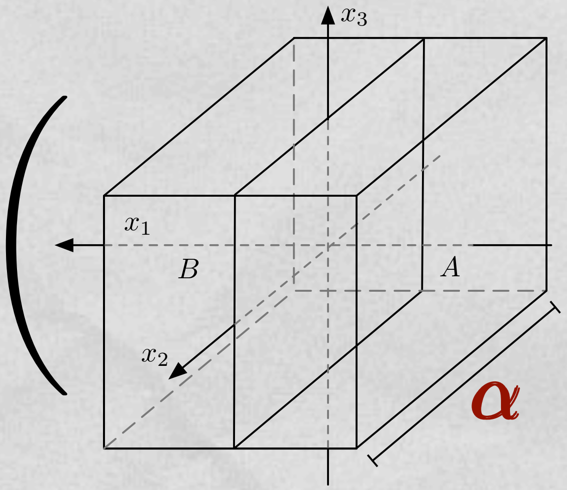
PHASE TRANSITIONS FOR THIN FILMS

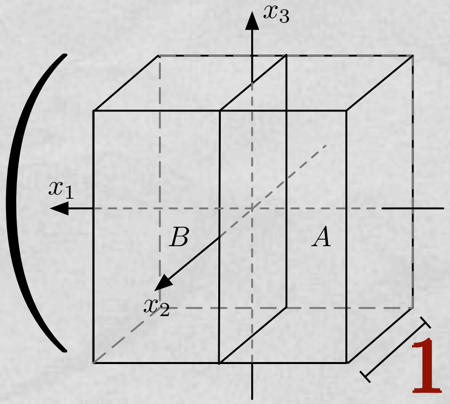
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 1. Γ- lim inf

Scaling Properties

$\Gamma - \liminf\limits_{\varepsilon \to 0^+} I_\varepsilon^1 \Bigg($ 

$\Bigg) = \boldsymbol{\alpha}\ \Gamma - \liminf\limits_{\varepsilon \to 0^+} I_\varepsilon^1 \Bigg($ 

$\Bigg)$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 1. Γ- lim inf

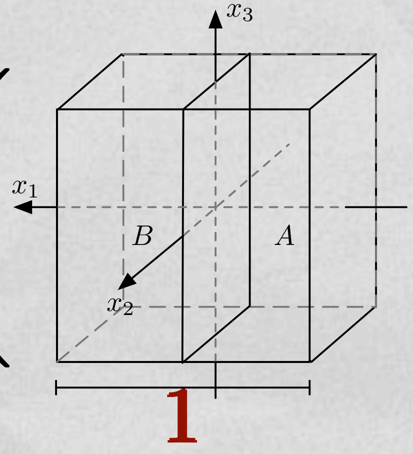
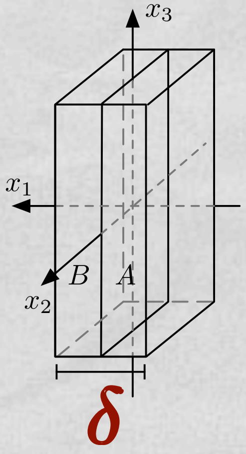
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 1. Γ- lim inf

Energy Concentrates on Discontinuity Surface

$\Gamma - \liminf\limits_{\varepsilon \to 0^+} I_\varepsilon^1 \Bigg($  $\Bigg) = \Gamma - \liminf\limits_{\varepsilon \to 0^+} I_\varepsilon^1 \Bigg($  $\Bigg)$

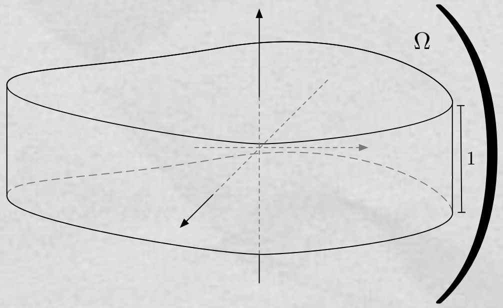
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

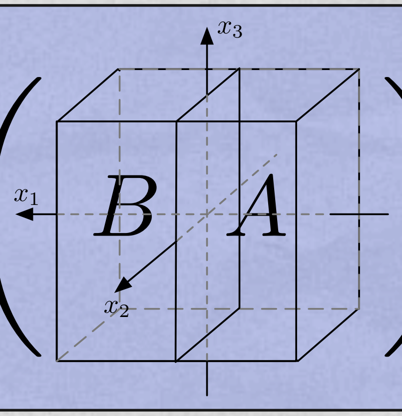
Idea of Proof

Step 1. Γ- lim inf

$$\boxed{\Gamma - \liminf_{\varepsilon \to 0^+} I^1_\varepsilon \left((u,b); \quad \right) \geqslant K^\star_\nu \, \mathcal{H}^1\big(S(\nabla' u, b)\big)}$$



$$\boxed{K^\star_\nu := \Gamma - \liminf_{\varepsilon \to 0^+} I^1_\varepsilon \left(\quad \right)}$$



PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$

$$A = \begin{array}{c} \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right) \\ \qquad A' \qquad\qquad A_3 \end{array}$$

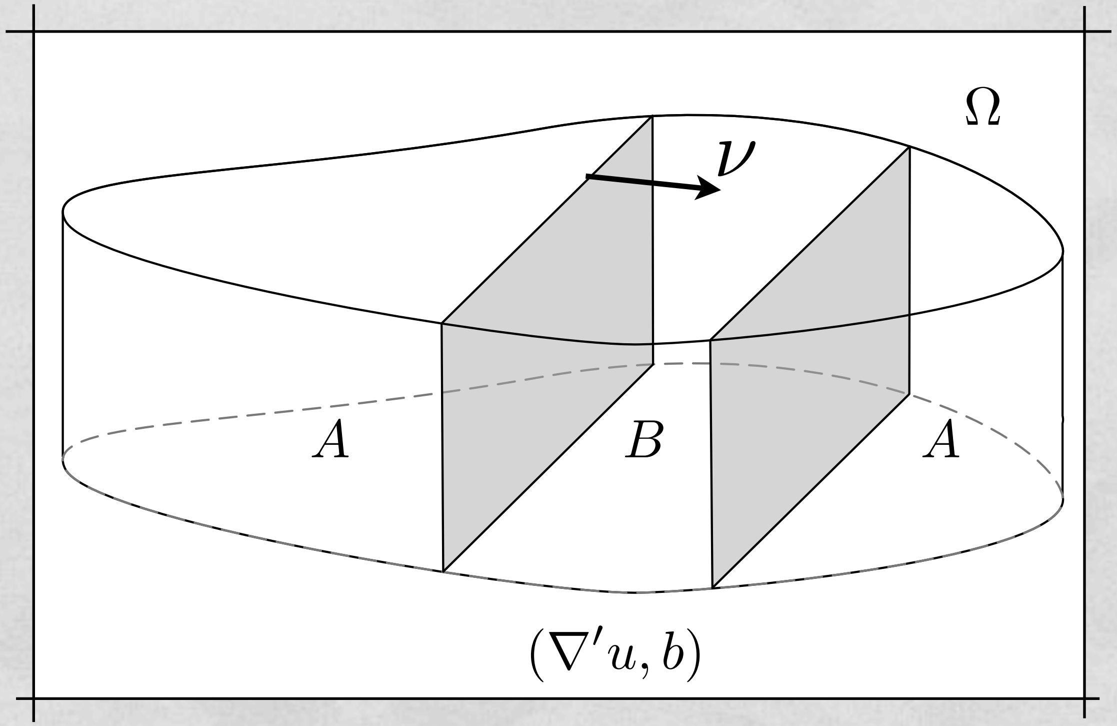
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$



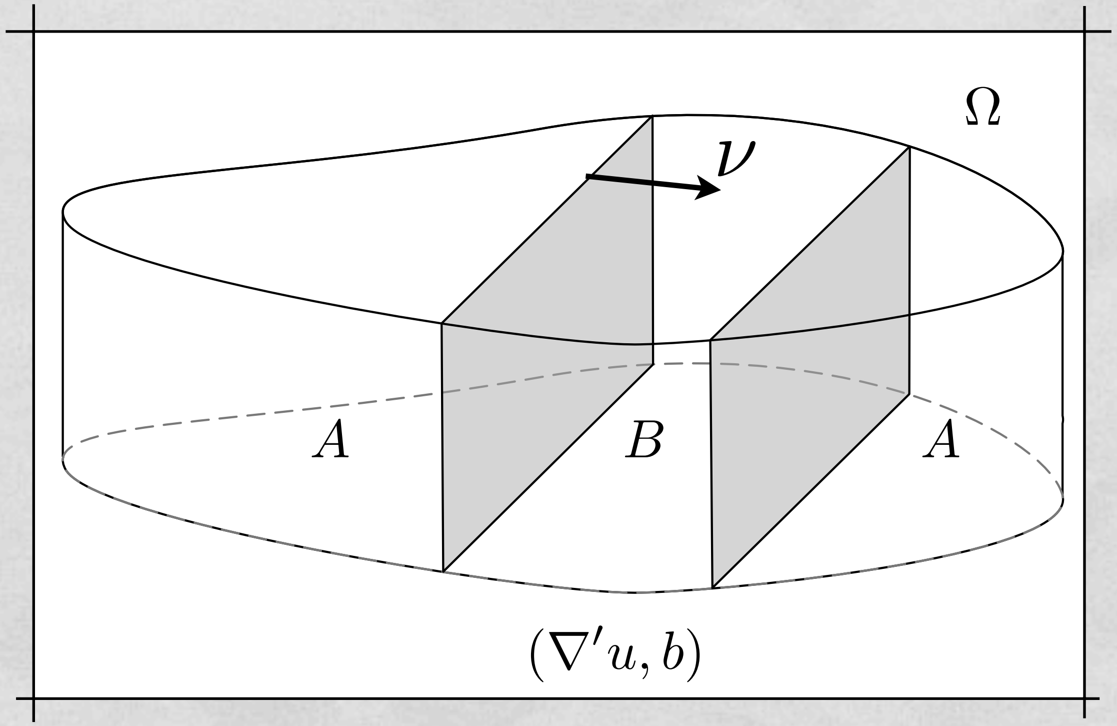
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$



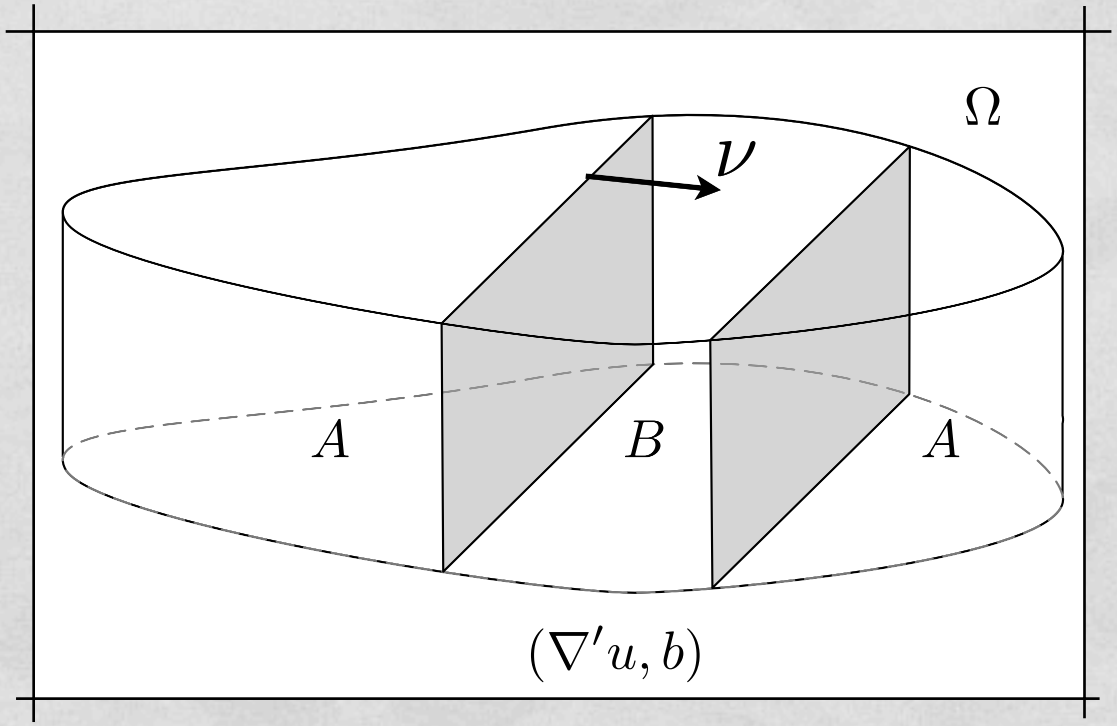
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$



$\boxed{(H_4) \quad \text{best path from } A' \text{ to } B' \text{ does not depend on } \nu^\perp}$

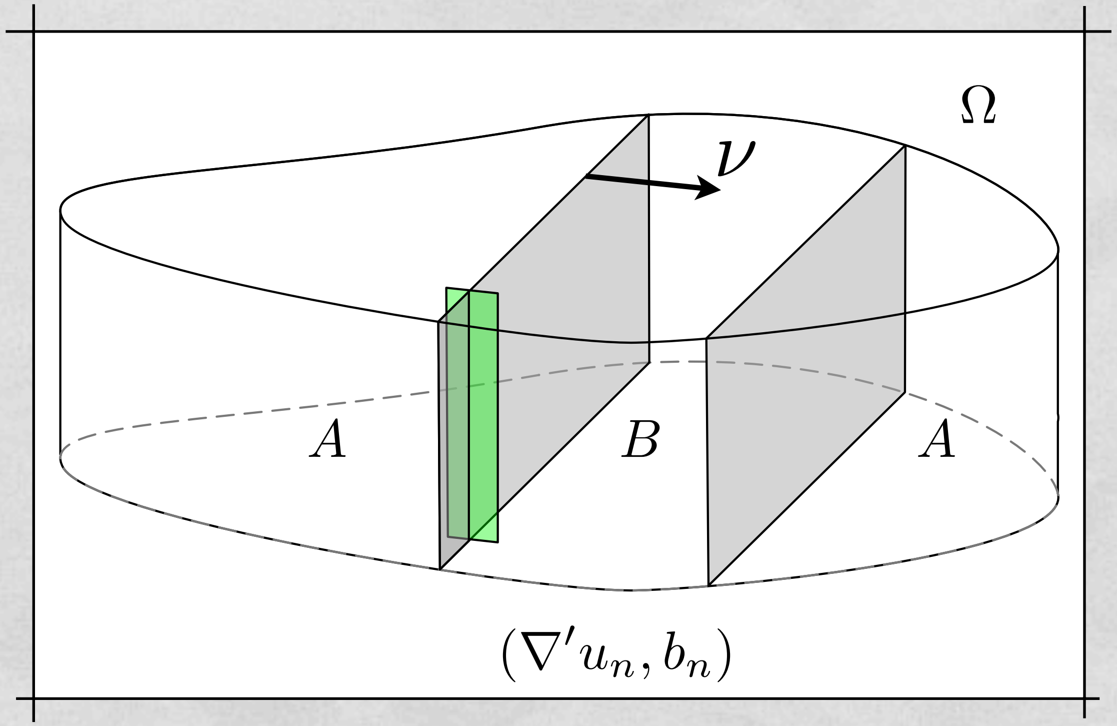
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$



Construct
Recovery Sequence

$$\boxed{(H_4) \quad \text{best path from } A' \text{ to } B' \text{ does not depend on } \nu^\perp}$$

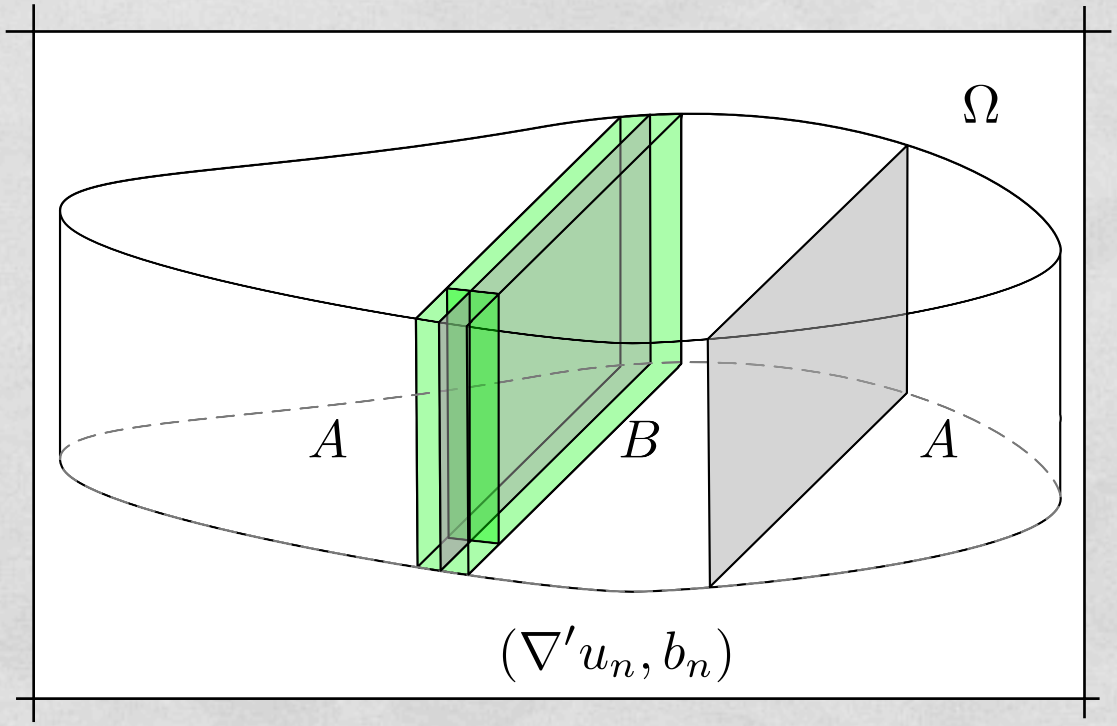
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$

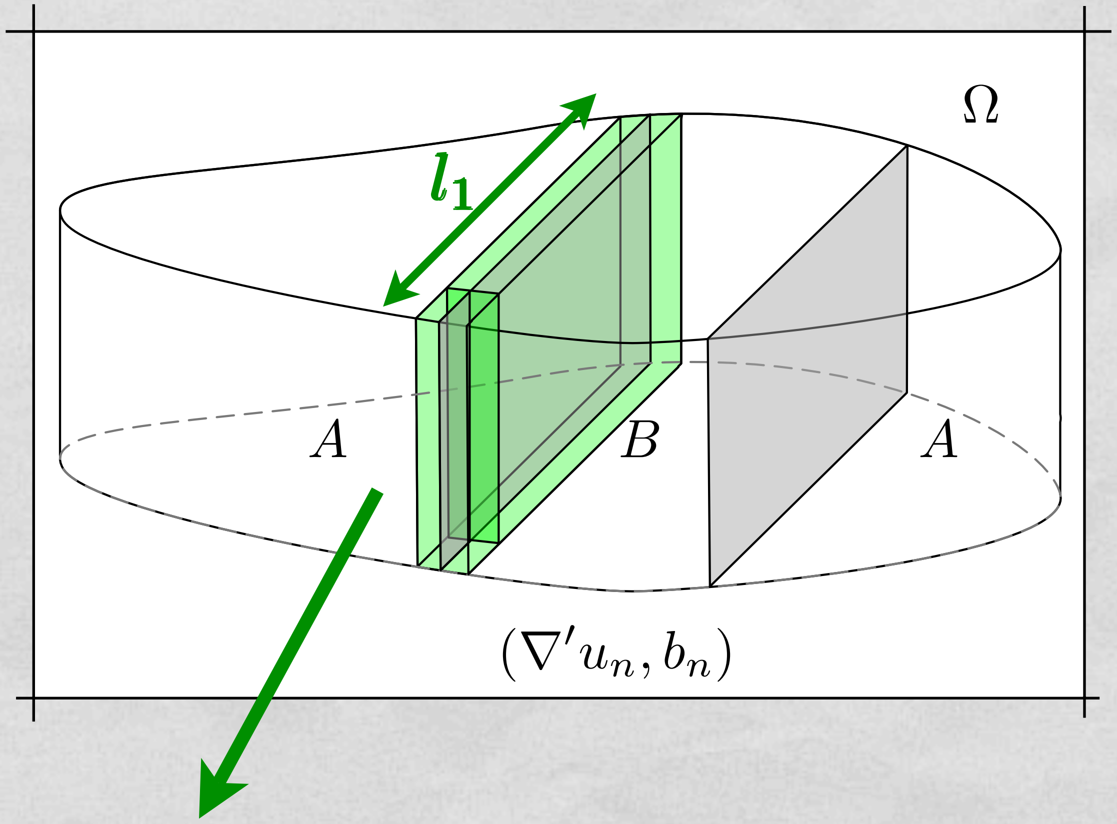


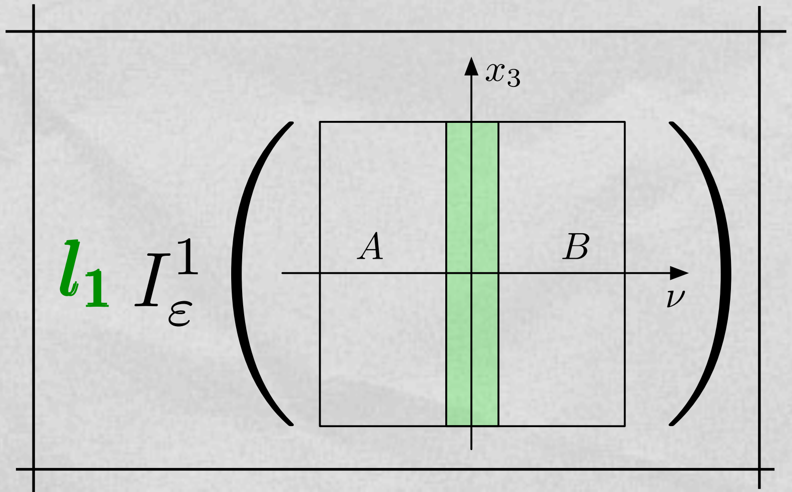
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$



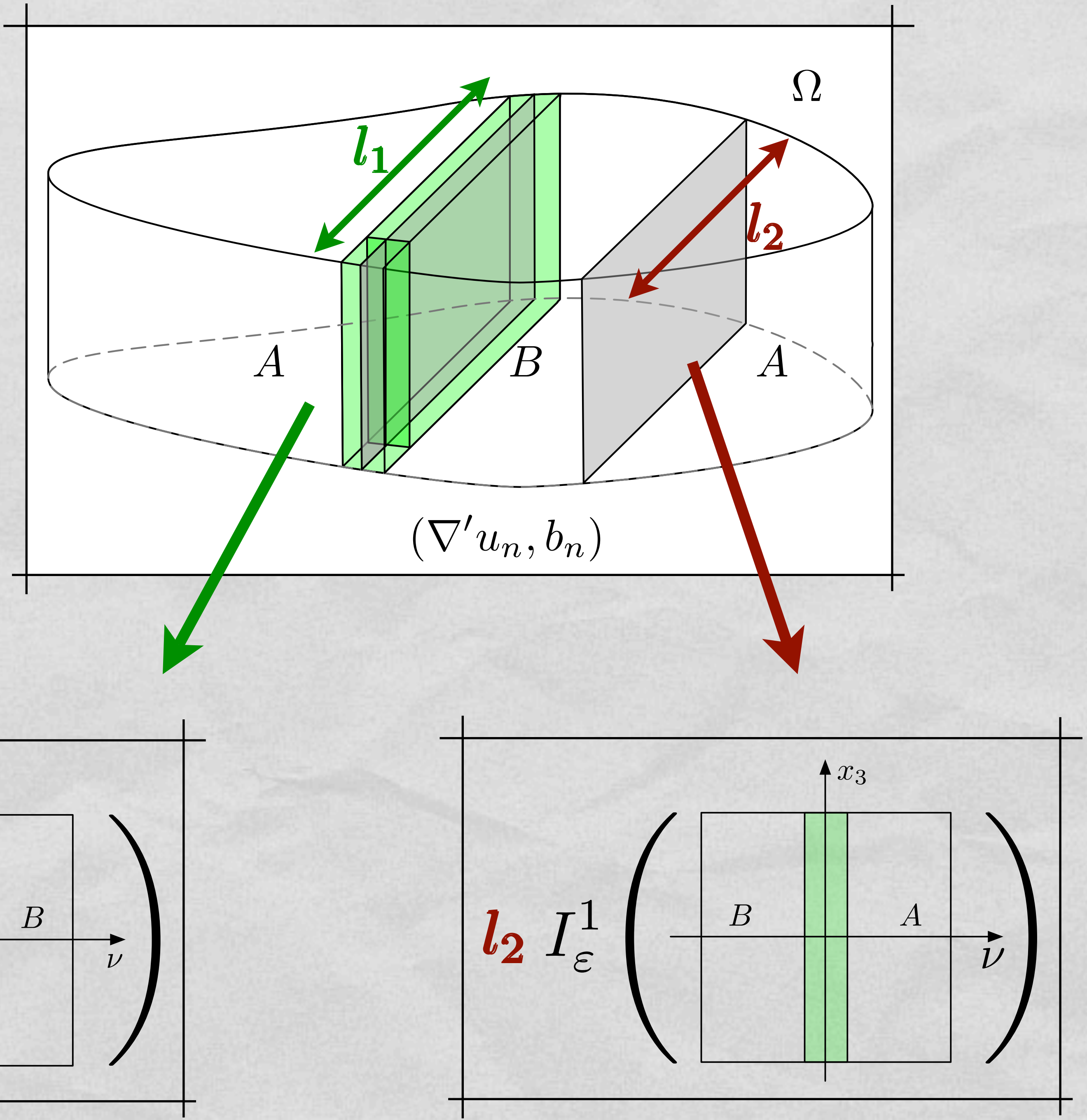


PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case Idea of Proof

Step 2. Γ- lim sup

Case 2a. $A' \neq B'$



PHASE TRANSITIONS FOR THIN FILMS

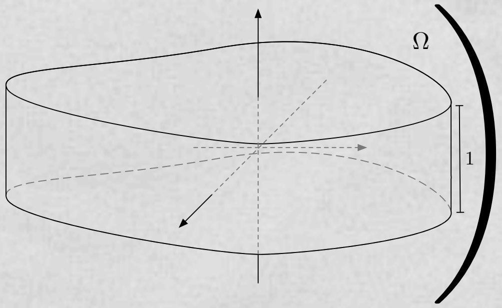
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

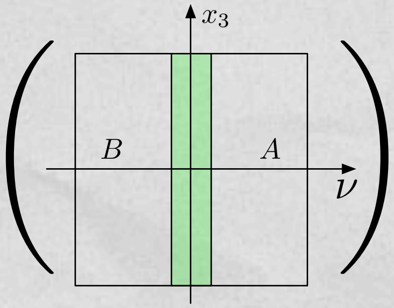
Step 2. Γ- lim sup

Case 2a. $A' \neq B'$

$I^1_\varepsilon \Bigg(u_\varepsilon;$



$\Bigg) \leqslant I^1_\varepsilon \Bigg($



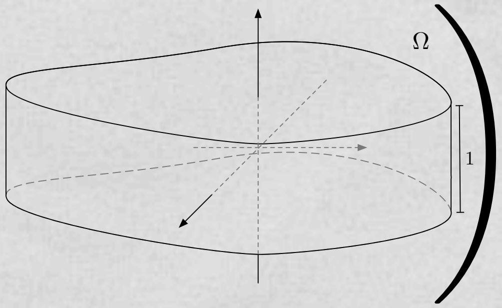
$\Bigg) \mathcal{H}^1 \big(S(\nabla' u, b) \big)$

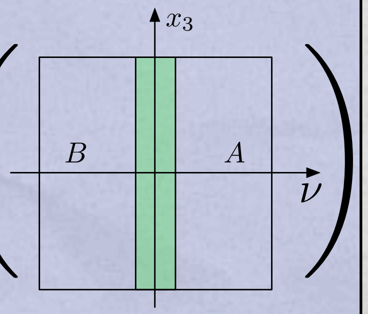
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case — Idea of Proof

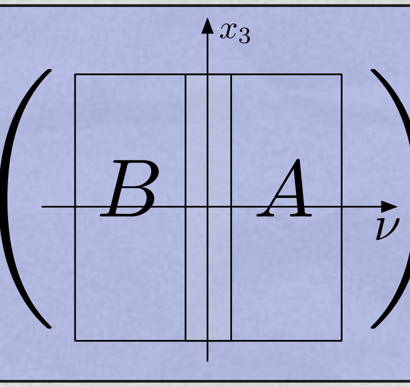
Step 2. Γ- lim sup

Case 2a. $A' \neq B'$

$I^1_\varepsilon \left(u_\varepsilon ; \right.$


$\left. \right) \leqslant I^1_\varepsilon \left(\right.$


$\left. \right) \mathcal{H}^1 \left(S(\nabla' u, b) \right)$

$K^\star_\nu := \Gamma - \liminf\limits_{\varepsilon \to 0^+} I^1_\varepsilon \left(\right.$


$\left. \right)$

PHASE TRANSITIONS FOR THIN FILMS

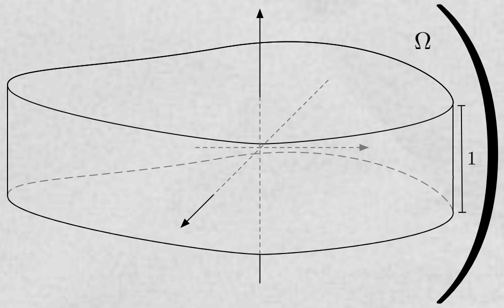
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

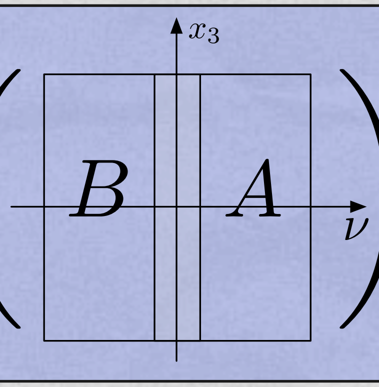
Case 2a. $A' \neq B'$

$$I^1_\varepsilon\left(u_\varepsilon;\right.$$



$$\left.\right) \leqslant \quad K^\star_\nu\, \mathcal{H}^1\big(S(\nabla' u, b)\big)$$

$$K^\star_\nu := \Gamma - \liminf_{\varepsilon \to 0^+} I^1_\varepsilon \Big($$



$$\Big)$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$

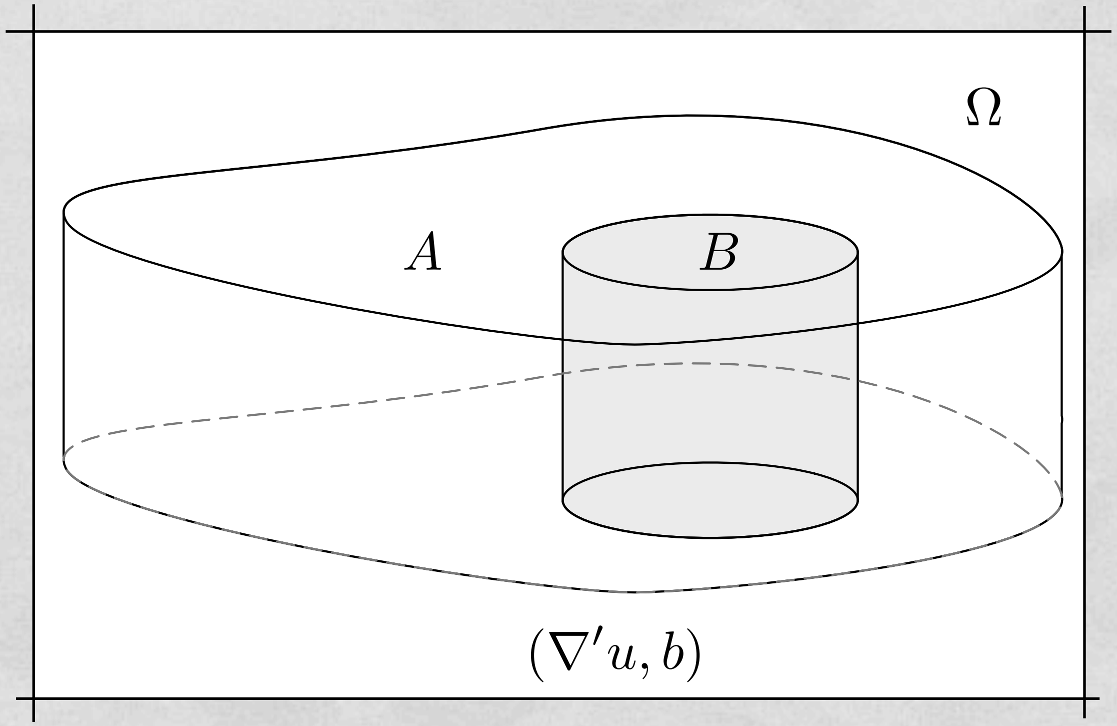
$$\begin{array}{cc} A = & \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right) \\ & \begin{array}{cc|c} \multicolumn{2}{c}{A'} & A_3 \end{array} \end{array}$$

PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$



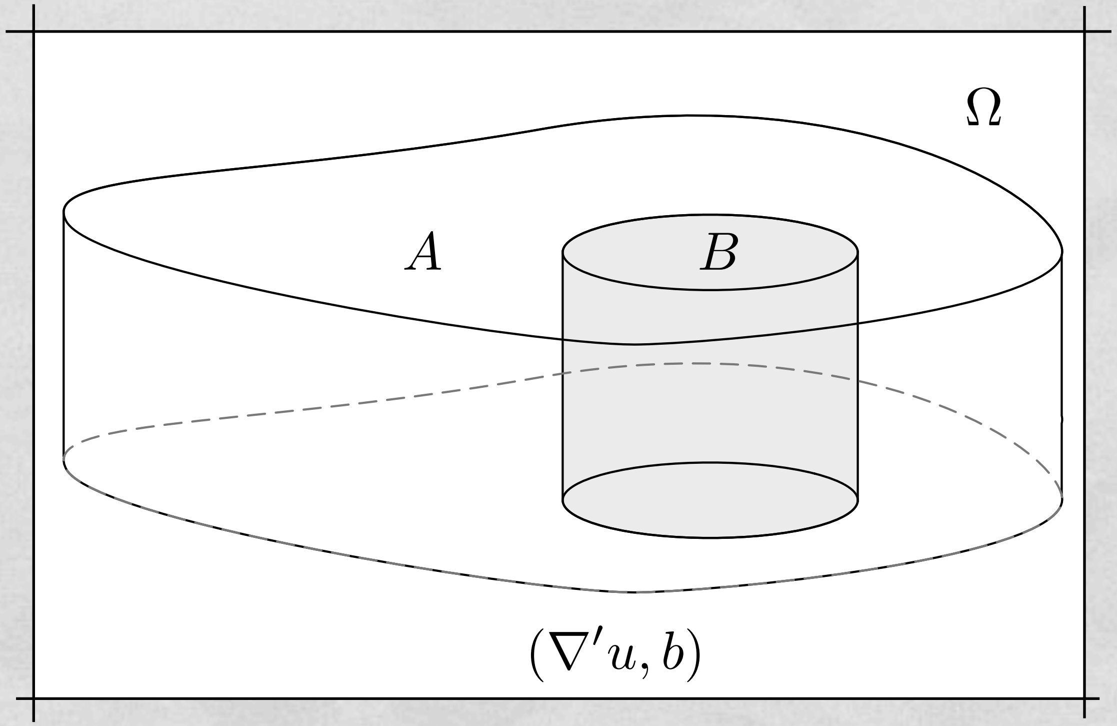
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$



$$\boxed{(H_4) \qquad W = W(|\xi'|, \xi_3)}$$

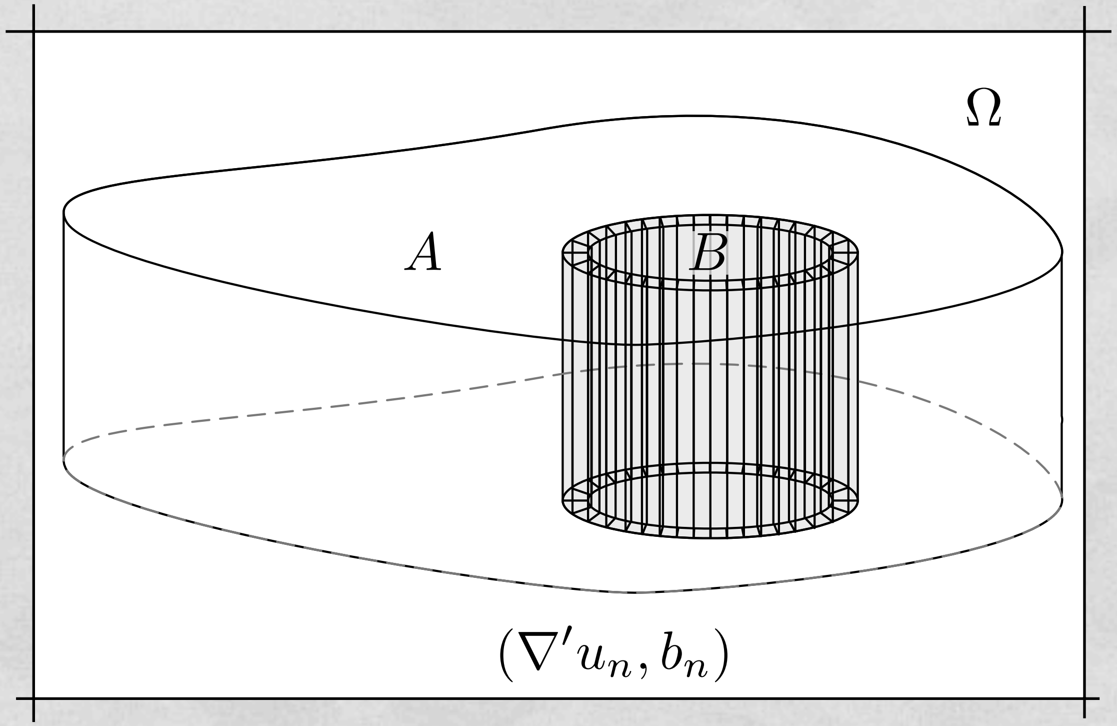
PHASE TRANSITIONS FOR THIN FILMS

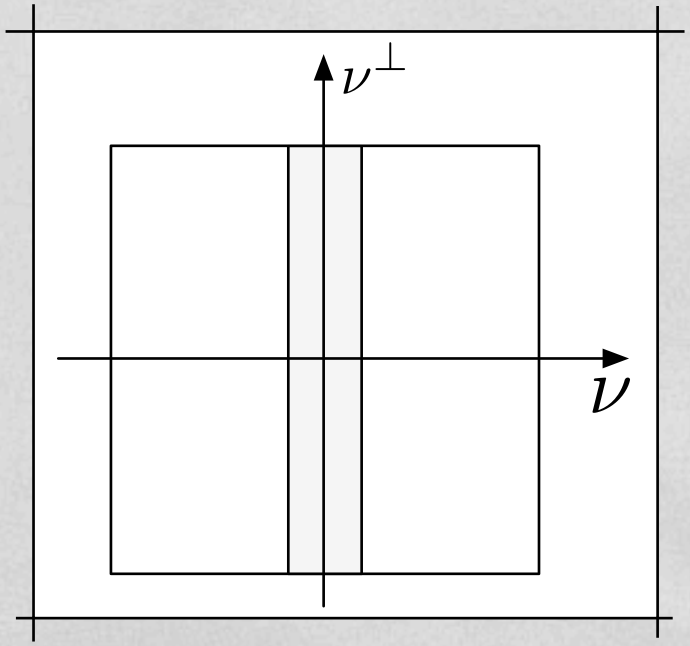
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$





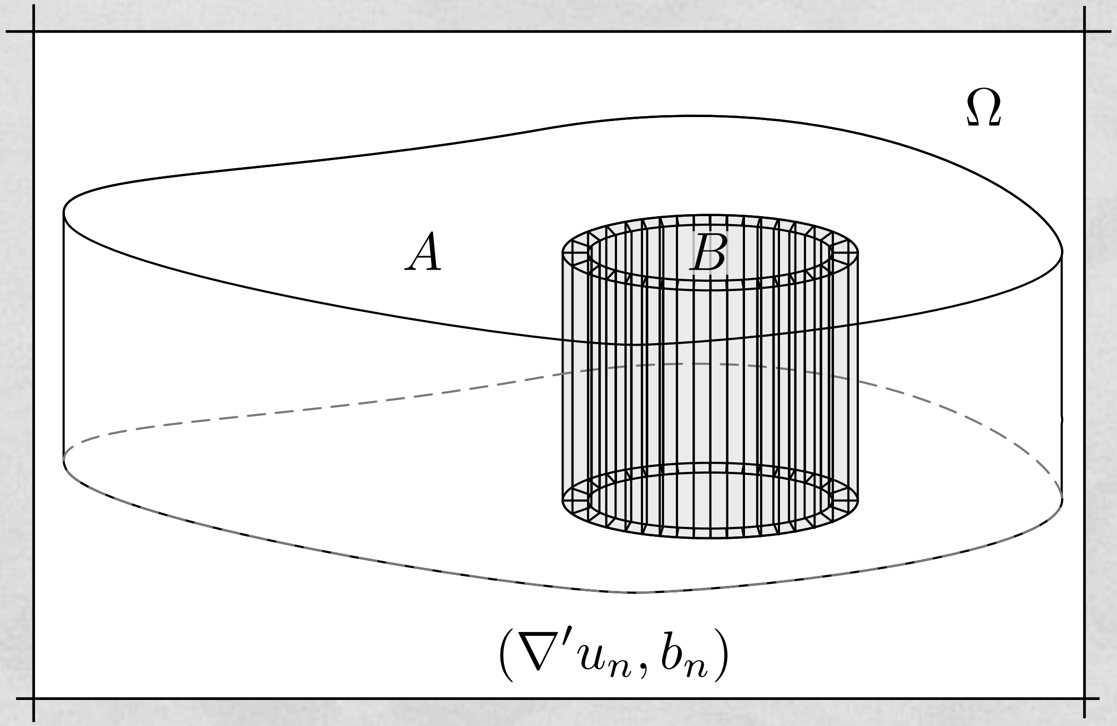
PHASE TRANSITIONS FOR THIN FILMS

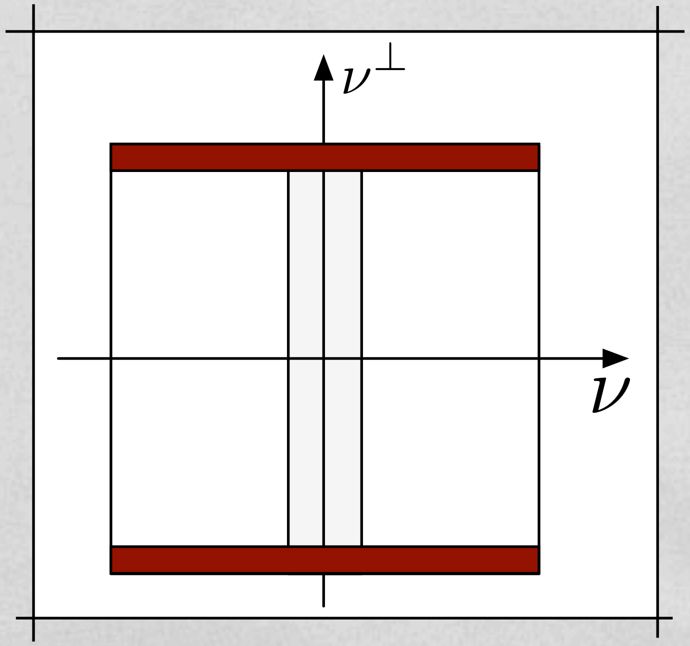
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$





Need to “glue” the cubes periodically on $\nu^\perp$

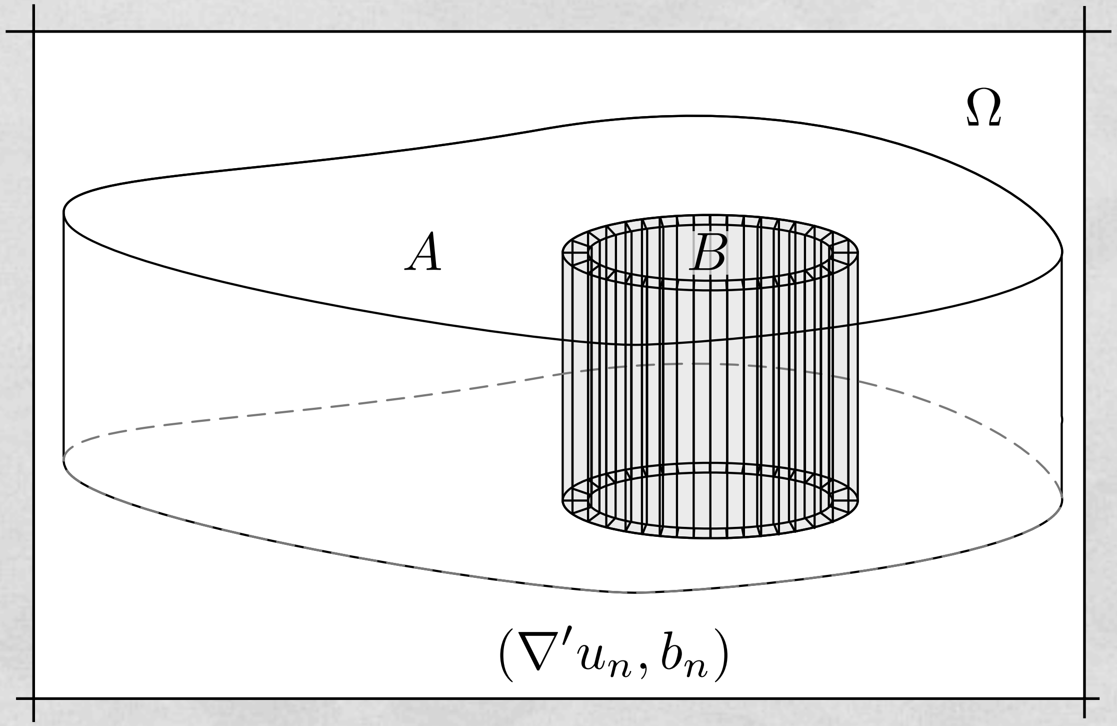
PHASE TRANSITIONS FOR THIN FILMS

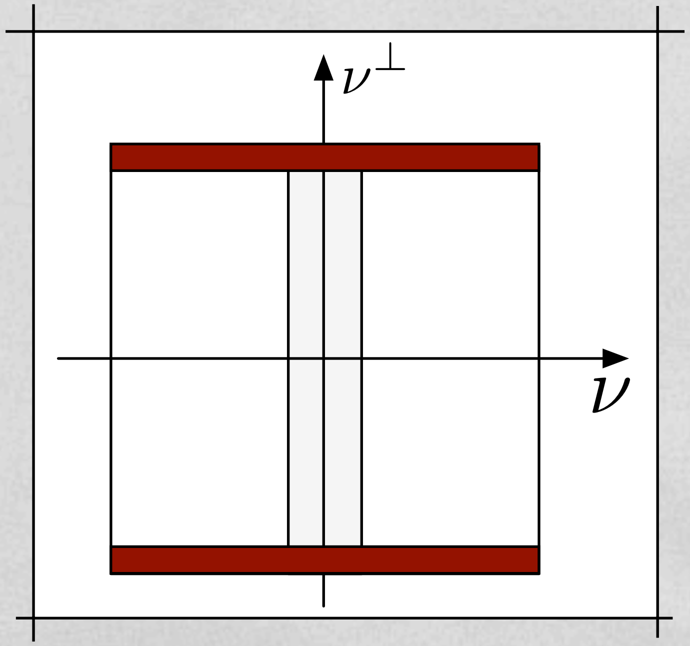
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$





Need to "glue" the cubes periodically on $\nu^\perp$

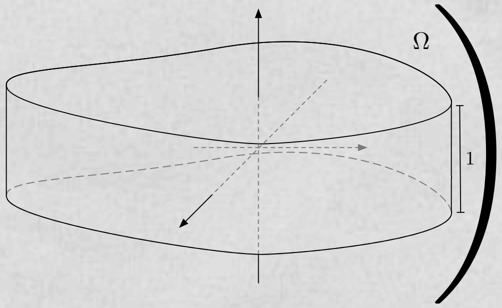
Use Scaling Properties of Energy

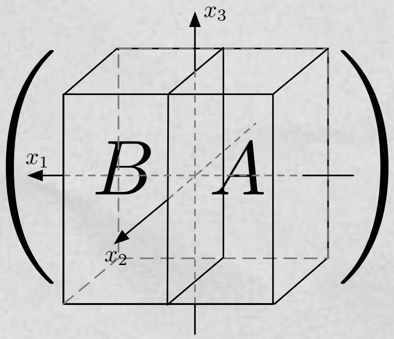
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$I^1_\varepsilon\Big((u,b);$ 

$\Big) \leqslant I^1_\varepsilon\Big($ 

$\Big)\ \mathcal{H}^1\big(S(\nabla' u, b)\big)$

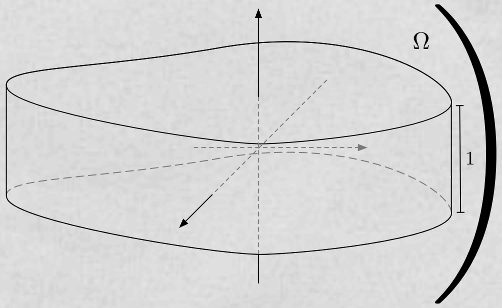
PHASE TRANSITIONS FOR THIN FILMS

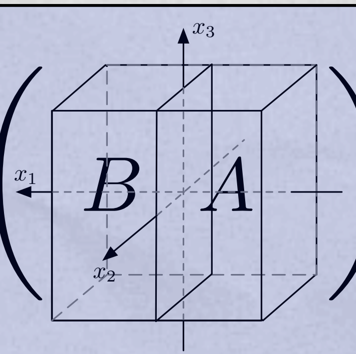
$\boxed{\gamma = 1}$ Critical Case

Idea of Proof

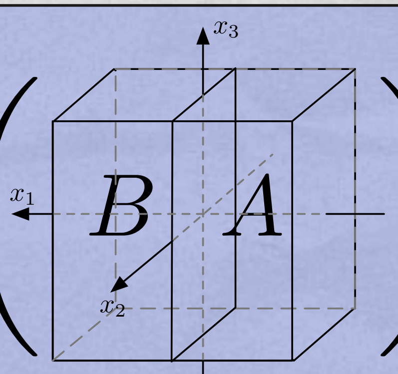
Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$I_\varepsilon^1 \Big((u, b);$


$\Big) \leqslant I_\varepsilon^1 \Big($


$\Big) \; \mathcal{H}^1 \big(S(\nabla' u, b) \big)$

$K_\nu^\star := \Gamma - \liminf_{\varepsilon \to 0^+} I_\varepsilon^1 \Big($


$\Big)$

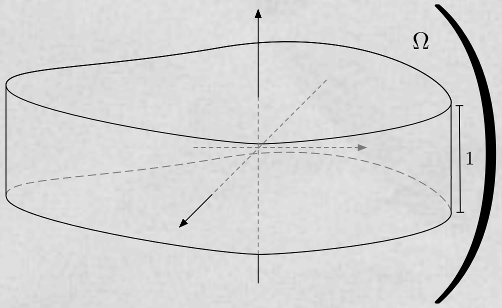
PHASE TRANSITIONS FOR THIN FILMS

$\boxed{\gamma = 1}$ Critical Case

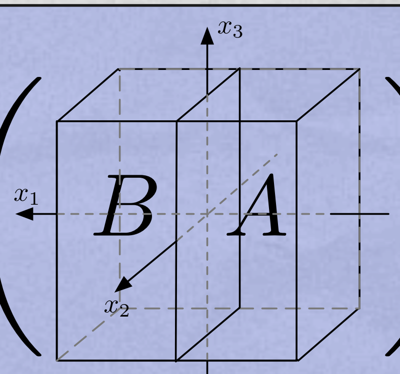
Idea of Proof

Step 2. Γ- lim sup

Case 2b. $A' = B', A_3 \neq B_3$

$I^1_\varepsilon\Bigg((u,b);$ 

$\Bigg) \leqslant K^\star_\nu\, \mathcal{H}^1\left(S(\nabla' u, b)\right)$

$K^\star_\nu := \Gamma - \liminf\limits_{\varepsilon\to0^+} I^1_\varepsilon\Bigg($ 

$\Bigg)$

PHASE TRANSITIONS FOR THIN FILMS

PHASE TRANSITIONS FOR THIN FILMS

Thank You

PHASE TRANSITIONS IN THIN FILMS

Outline of the Talk

- Historic Context
  - for Phase Transitions
  - for Thin Films
- Γ- limit

- Phase Transitions in Thin Films
  - Hypotheses on W
  - Different Regimes
  - Rigidity
  - Critical Case
  - Subcritical Case
  - Supercritical Case
    - Γ- lim inf
    - Γ- lim sup
      - Case $A' \neq B'$
      - Case $A' = B', A_3 \neq B_3$