higher-order phase transitions with line-tension effect

bernardo galvão-sousa

carnegie mellon university

historic context

classical theory for phase transitions for non-interacting fluid

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Ω container

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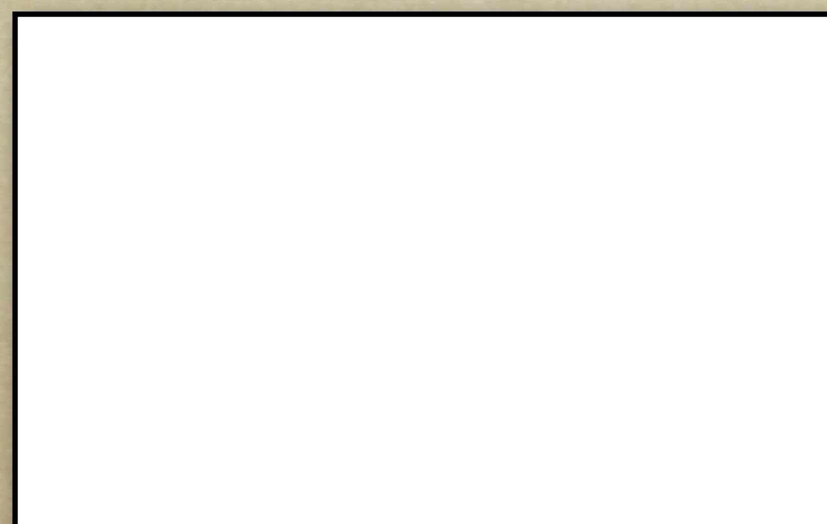

Ω container

u fluid density satisfies

mass: $\displaystyle\int_\Omega u\,dx = V$

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Ω container

u fluid density satisfies

$$\min_{\substack{u \in L^1(\Omega) \\ \int_\Omega u\, dx = V}} \int_\Omega W(u)\, dx$$

$$a|\Omega| < V < b|\Omega|$$

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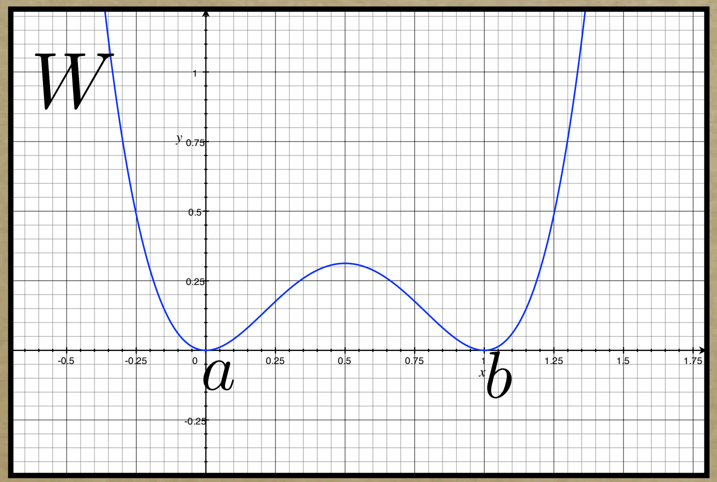

W = Gibbs free energy density

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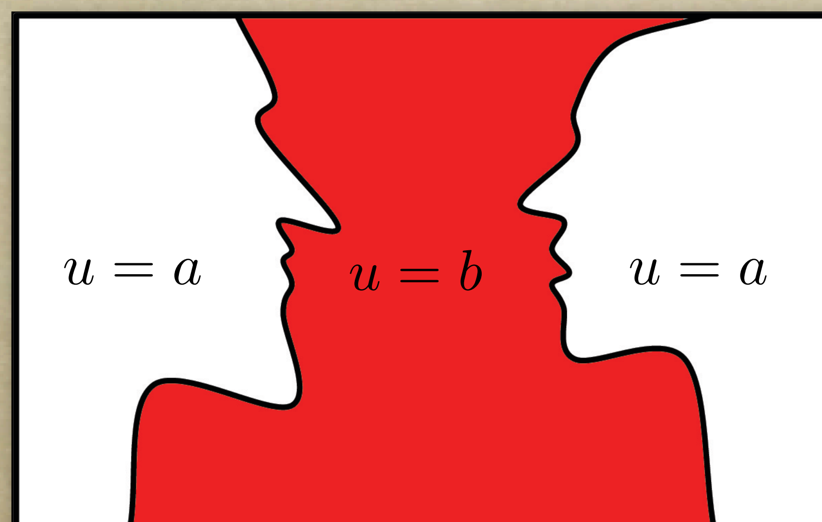

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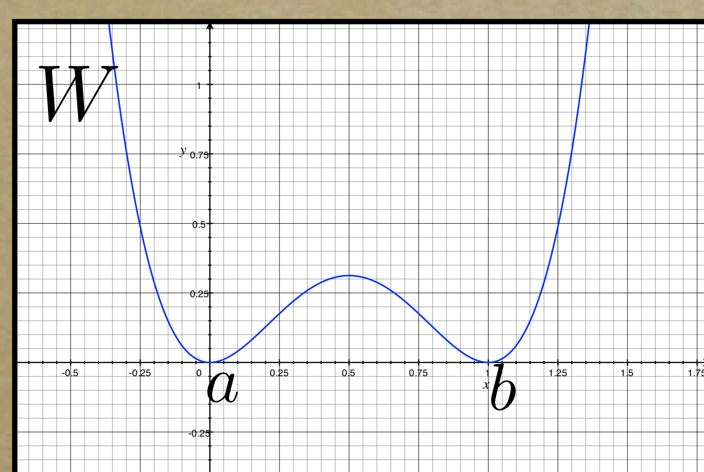

W = Gibbs free energy density

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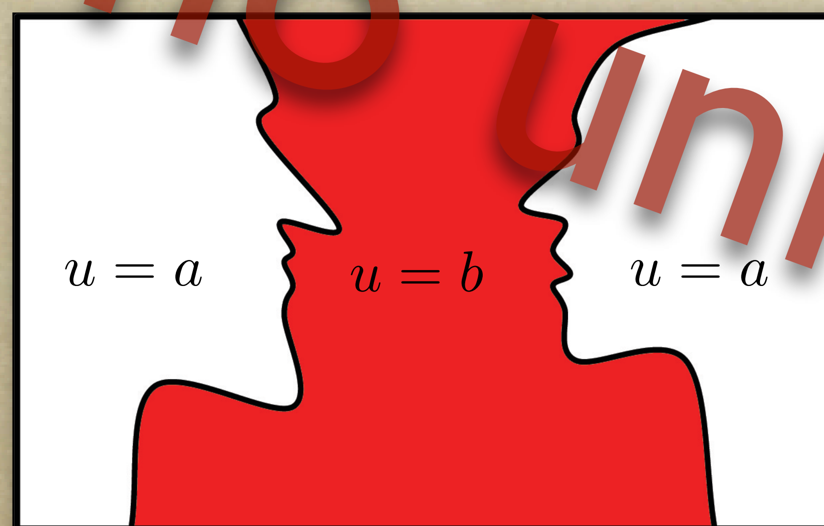
no uniqueness



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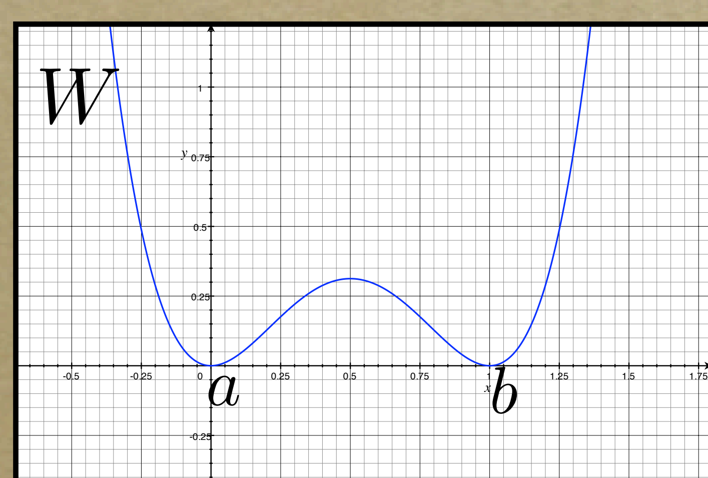

W = Gibbs free energy density

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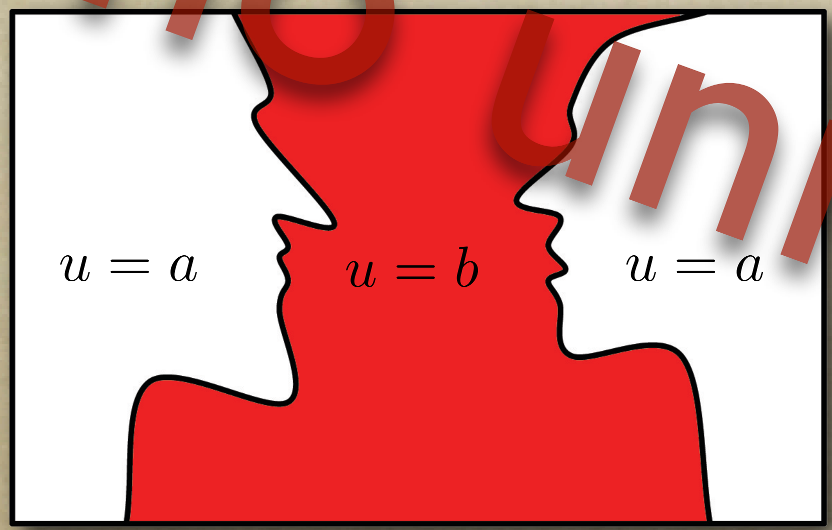

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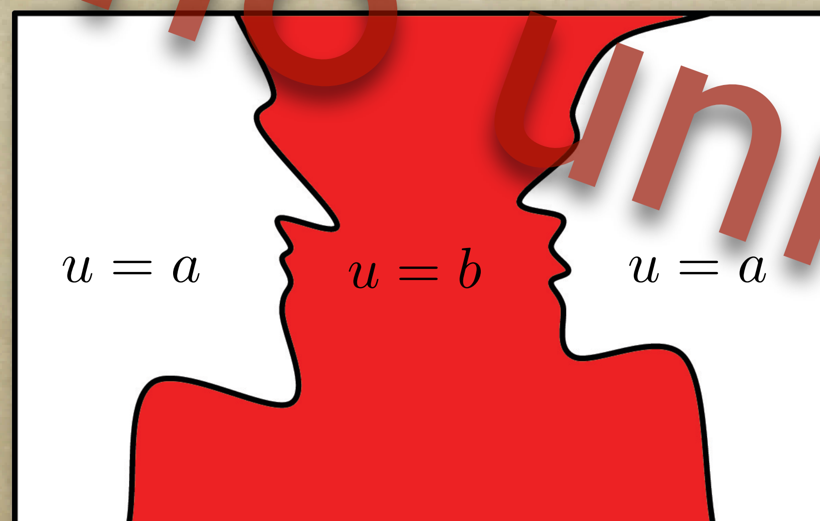

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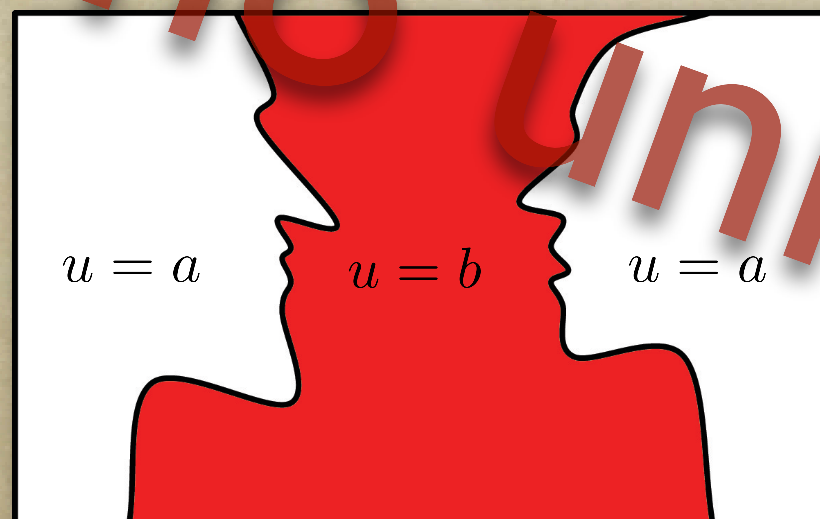

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no uniqueness

Van der Waals-Cahn-Hilliard

+ perturbation ⇨ selection criterium

$$\min_{\substack{u \in L^1(\Omega) \\ \int_\Omega u\,dx = V}} \left\{ \varepsilon^{2k} \int_\Omega |D^k u|^2\,dx + \int_\Omega W(u)\,dx \right\}$$

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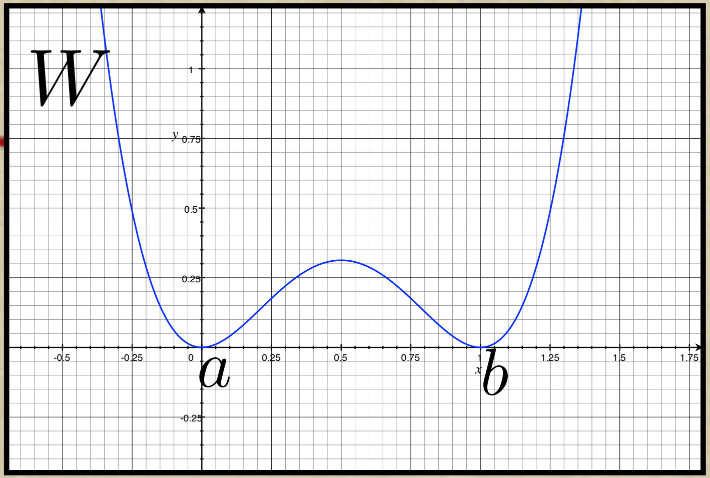

Gurtin '85

$$\varepsilon \int_\Omega |Du|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx$$

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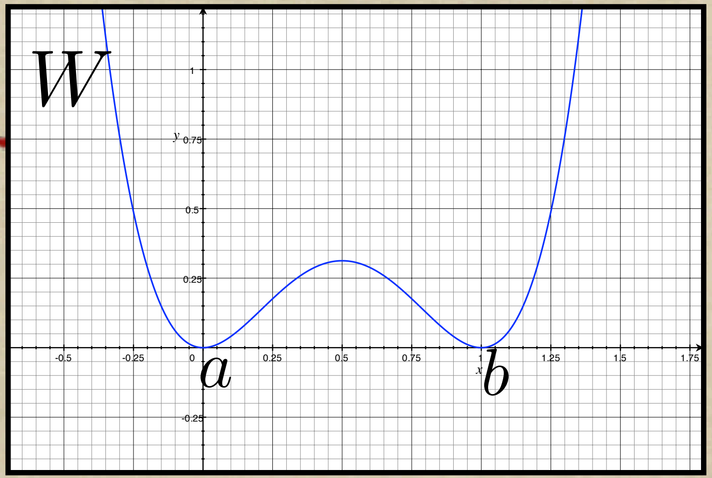

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conjecture

minimizer u_ε

$$u_\varepsilon \xrightarrow{\varepsilon \to 0^+} u_0$$

u_0 physically-preferred solution

minimizes interface area

$$\mathrm{Per}_\Omega(\{u_0 = a\}) = \min \mathrm{Per}_\Omega(\{u = a\})$$

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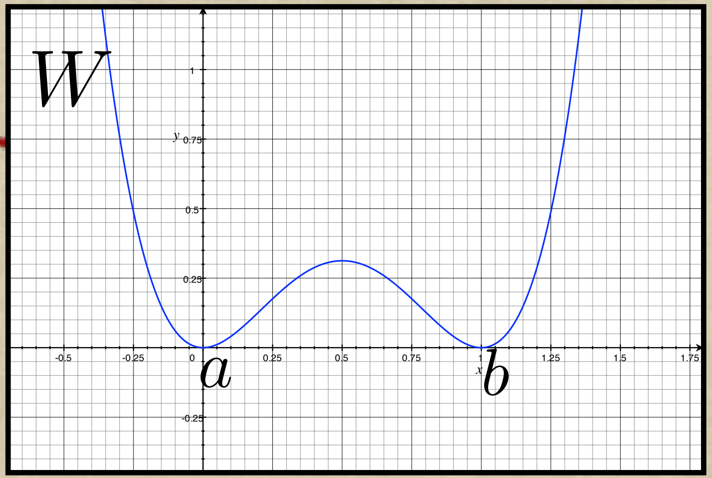

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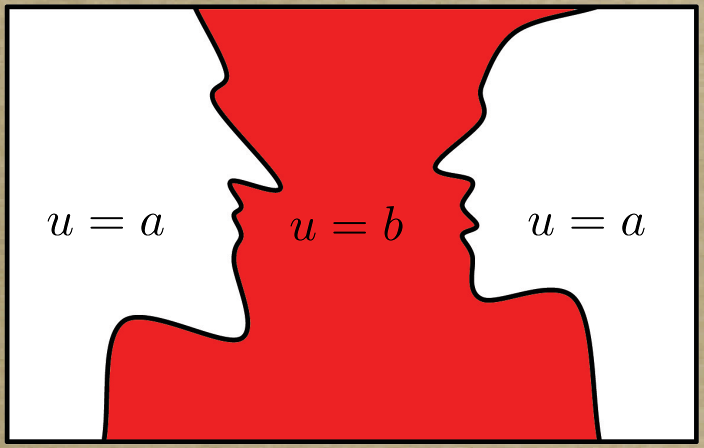

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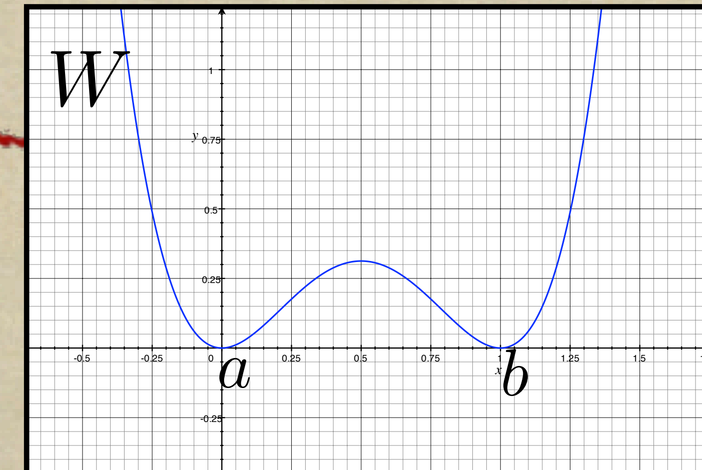

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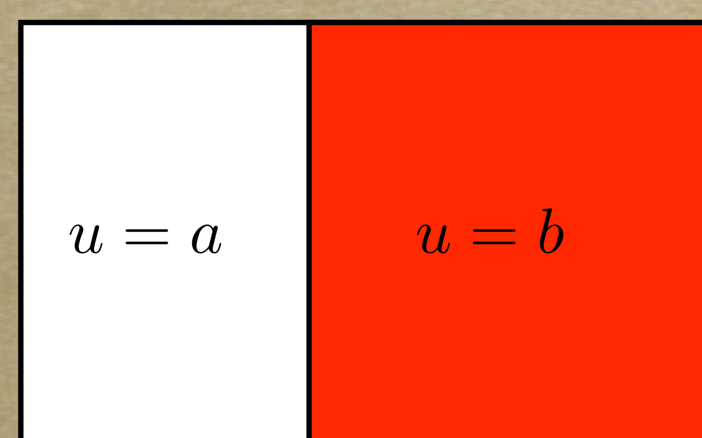

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Modica, Mortola '77

$$\varepsilon \int_{\mathbb{R}^N} |Du|^2 \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \sin^2\left(\frac{\pi u}{\varepsilon}\right) \, dx$$

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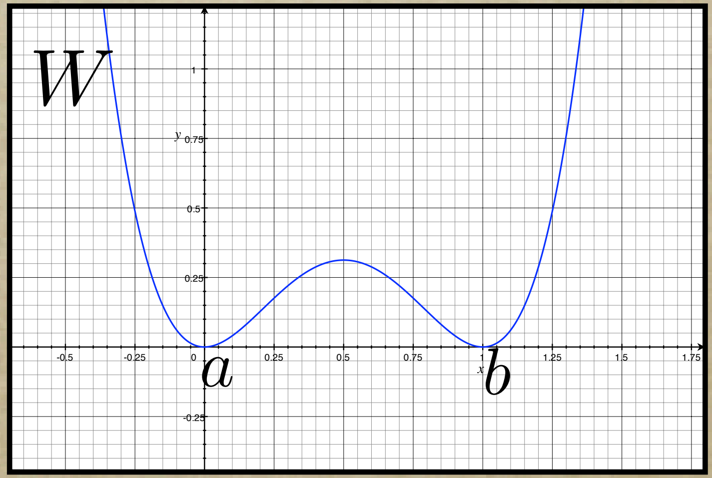
Modica '87, Sternberg '88

$$\varepsilon \int_{\Omega} |Du|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) \, dx$$



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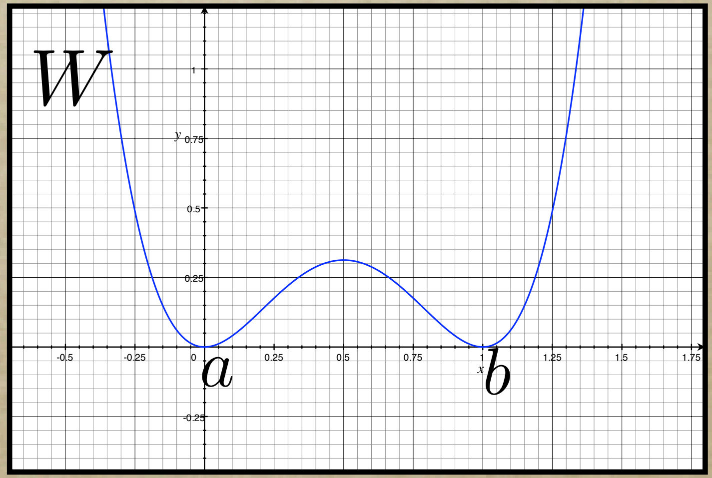

Modica '87

$$\varepsilon \int_{\Omega} |Du|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) \, dx + \varepsilon \int_{\partial\Omega} \sigma(Tu) \, d\mathcal{H}^{N-1}$$

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Alberti, Bouchitté, Seppecher '98

$$\varepsilon \int_\Omega |Du|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

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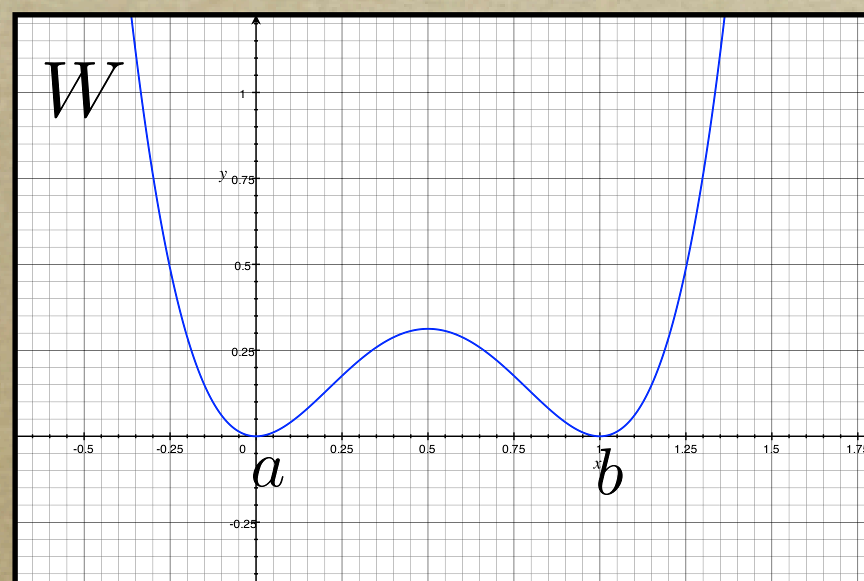

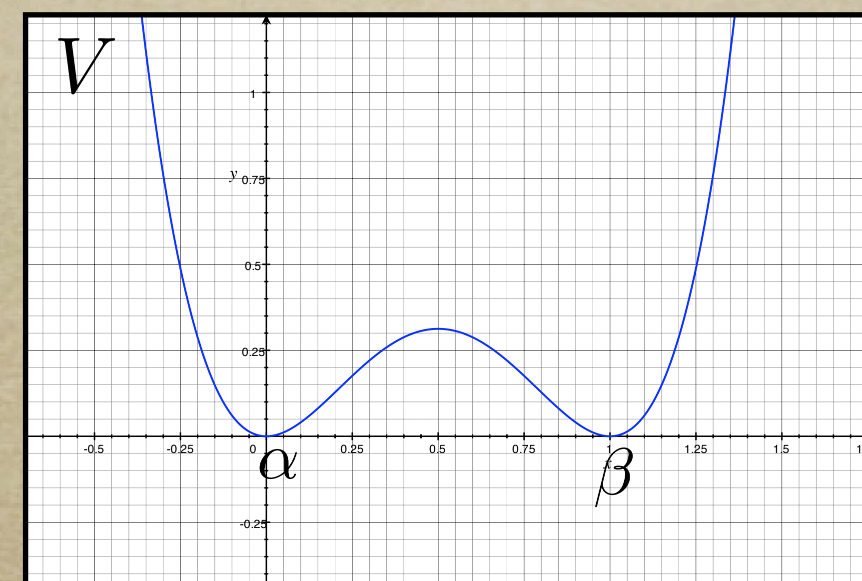

$$\lambda_\varepsilon \xrightarrow{\varepsilon \to 0^+} \infty$$

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Alberti, Bouchitté, Seppecher '98

$$\varepsilon \int_\Omega |Du|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}$$

critical regime: $\varepsilon \log \lambda_\varepsilon \longrightarrow L \in (0,\infty)$

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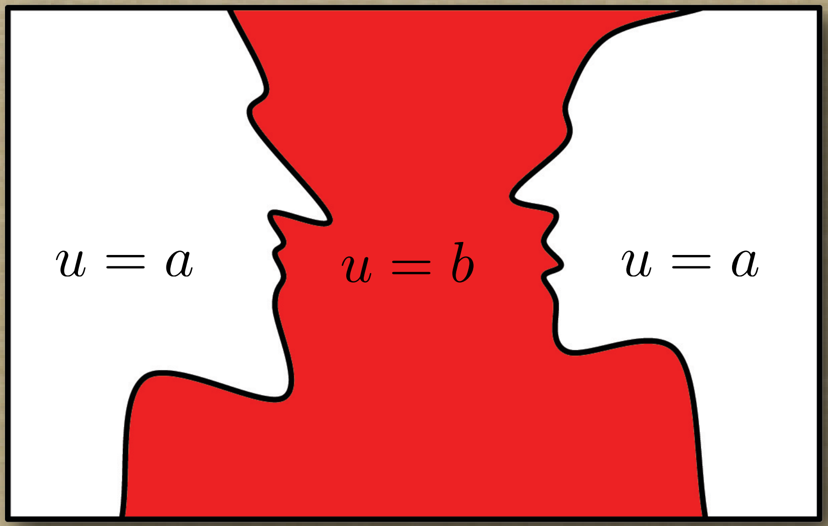
critical regime: $\varepsilon \log \lambda_\varepsilon \longrightarrow L \in (0,\infty)$

$$u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega;\{a,b\})$$



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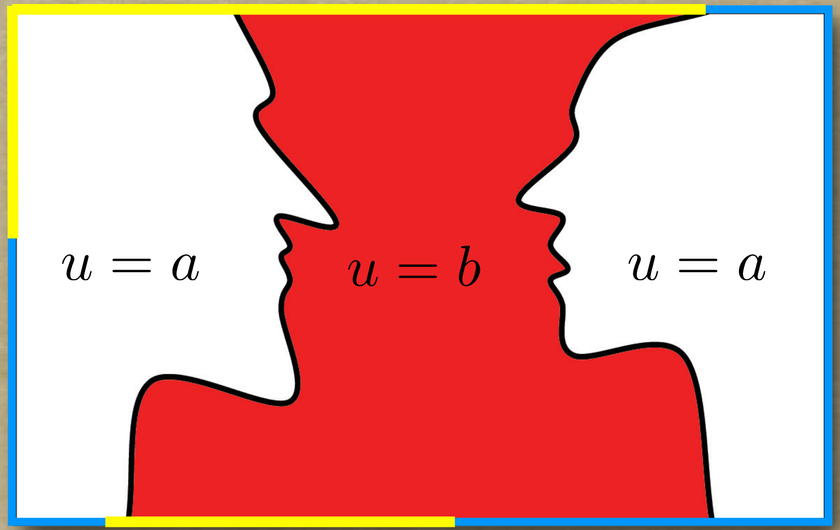
Alberti, Bouchitté, Seppecher '98

$$\varepsilon \int_\Omega |Du|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}$$

critical regime: $\varepsilon \log \lambda_\varepsilon \longrightarrow L \in (0,\infty)$

$$u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega;\{a,b\})$$
$$Tu_\varepsilon \xrightarrow{L^1} v \in BV(\partial\Omega;\{\alpha,\beta\})$$



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Alberti, Bouchitté, Seppecher '98

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critical regime: $\varepsilon \log \lambda_\varepsilon \longrightarrow L \in (0,\infty)$

Γ-limit

$$\mathcal{F}_\varepsilon \xrightarrow{\Gamma} \big|H(b) - H(a)\big| \mathrm{Per}_\Omega\big(\{u = a\}\big) + \int_{\partial\Omega} \big|H(Tu) - H(v)\big| d\mathcal{H}^{N-1} + (\beta-\alpha)^2 \frac{L}{\pi} \mathrm{Per}_{\partial\Omega}\big(\{v = \alpha\}\big)$$

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Alberti, Bouchitté, Seppecher '98

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$$H(x) := 2 \int_0^x \sqrt{W(t)}\, dt$$

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Fonseca, Mantegazza '00

$$\varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx$$

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Fonseca, Mantegazza '00

$$\varepsilon^3 \int_\Omega |D^2 u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx$$

Conti, Fonseca, Leoni '02

$$\varepsilon \int_\Omega |D^2 u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(Du)\, dx$$

Yu, Li, Ying '03

$$\varepsilon^{2k-1} \int_\Omega |D^k u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx$$

Monti, Serra Cassano '04

$$\varepsilon \int_\Omega q(x, Du)\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx$$

Garroni, Palatucci

$$\varepsilon^{p-2} \iint_{\Omega\times\Omega} \left|\frac{u(x)-u(y)}{x-y}\right|^p dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx$$

problem

$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}$$

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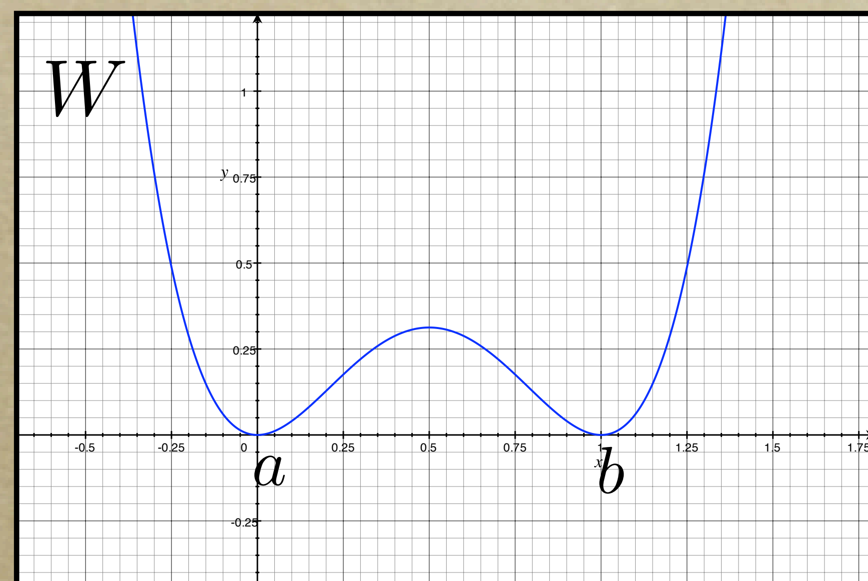

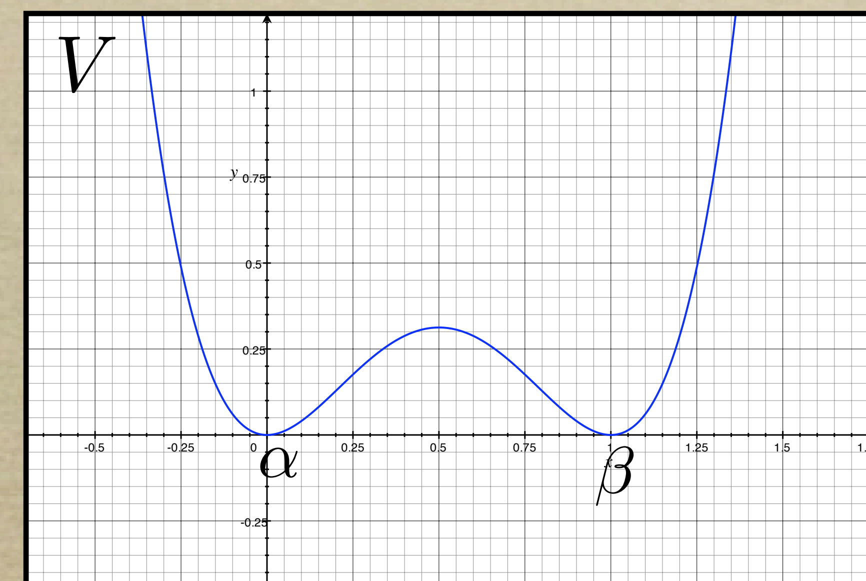

$$W(z) \geqslant C|z|^2 - \frac{1}{C}$$

$$V(z) \geqslant C|z|^2 - \frac{1}{C}$$

$$\lambda_\varepsilon \xrightarrow{\varepsilon \to 0^+} \infty$$

identify regimes (λ_ε vs ε)

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$$\varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

identify regimes (λ_ε vs ε)

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interior

$$\varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

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$$\varepsilon^3 \int_\Omega |D^2 u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\, d\mathcal{H}^{N-1}$$

Fonseca, Mantegazza '00

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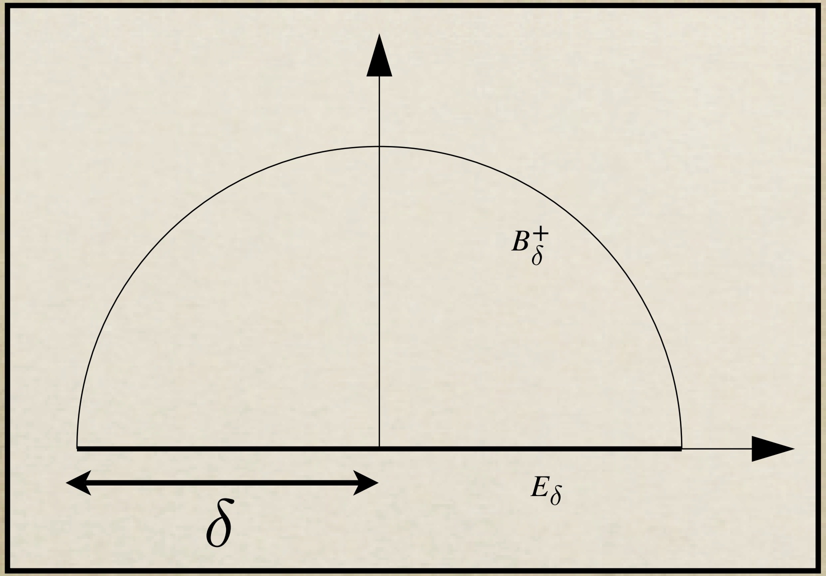
boundary



$$\varepsilon^3 \int_{B_\delta^+} |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_{B_\delta^+} W(u) \, dx + \lambda_\varepsilon \int_{E_\delta} V(Tu) \, d\mathcal{H}^{N-1}$$

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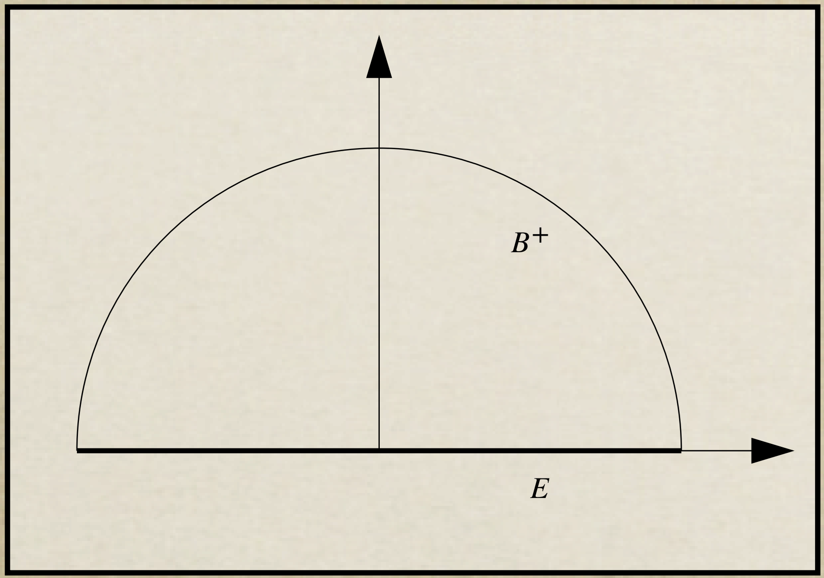

$$\varepsilon^3 \int_{B_\delta^+} |D^2 u|^2\, dx + \frac{1}{\varepsilon} \int_{B_\delta^+} W(u)\, dx + \lambda_\varepsilon \int_{E_\delta} V(Tu)\, d\mathcal{H}^{N-1}$$

$$\frac{\varepsilon^3}{\delta^2} \int_{B+} |D^2 u|^2\, dx + \frac{\delta^2}{\varepsilon} \int_{B+} W(u)\, dx + \lambda_\varepsilon \delta \int_E V(Tu)\, d\mathcal{H}^{N-1}$$

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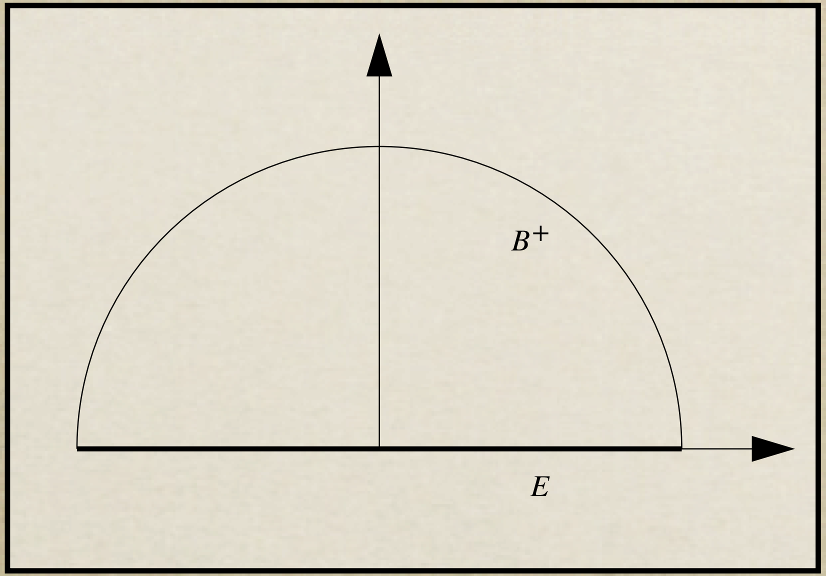

equi-partition

$$\frac{\varepsilon^3}{\delta^2}\int_{B^+}|D^2u|^2\,dx+\frac{\delta^2}{\varepsilon}\int_{B^+}W(u)\,dx+\lambda_\varepsilon\delta\int_E V(Tu)\,d\mathcal{H}^{N-1}$$

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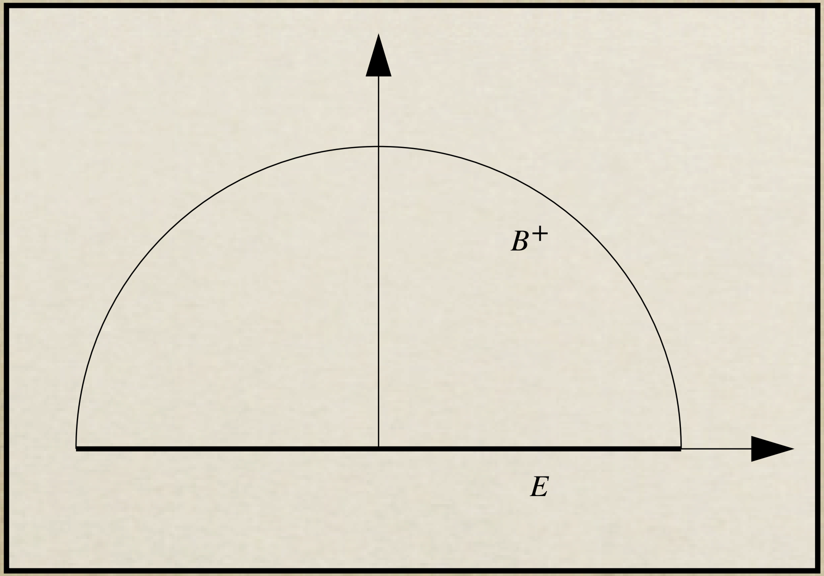

$$\frac{\varepsilon^3}{\delta^2}\int_{B^+}|D^2u|^2\,dx + \frac{\delta^2}{\varepsilon}\int_{B^+}W(u)\,dx + \lambda_\varepsilon\delta\int_E V(Tu)\,d\mathcal{H}^{N-1}$$

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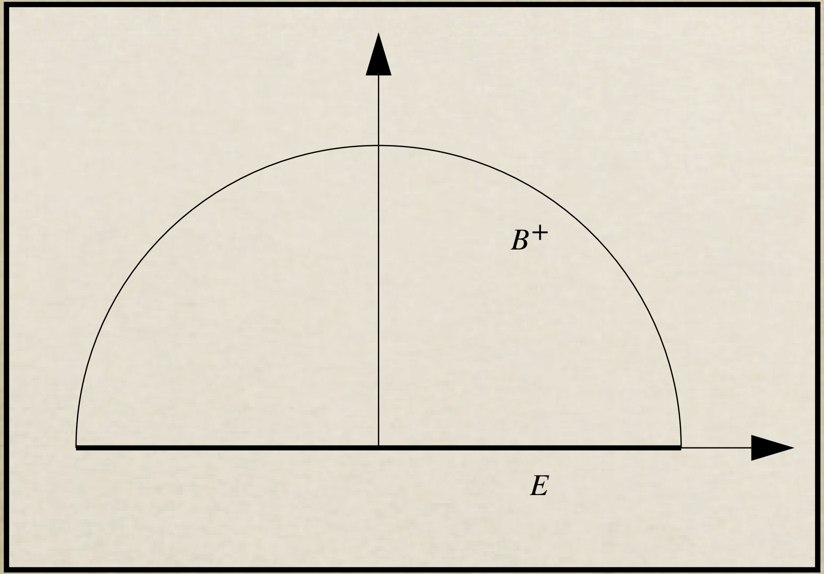

equi-partition

$$\frac{\varepsilon^3}{\delta^2}\int_{B+}|D^2u|^2\,dx+\frac{\delta^2}{\varepsilon}\int_{B+}W(u)\,dx+\lambda_\varepsilon\delta\int_E V(Tu)\,d\mathcal{H}^{N-1}$$

boundary

$$\delta\approx\varepsilon\lambda_\varepsilon^{-\frac{1}{3}}$$

identify regimes (λ_ε vs ε)

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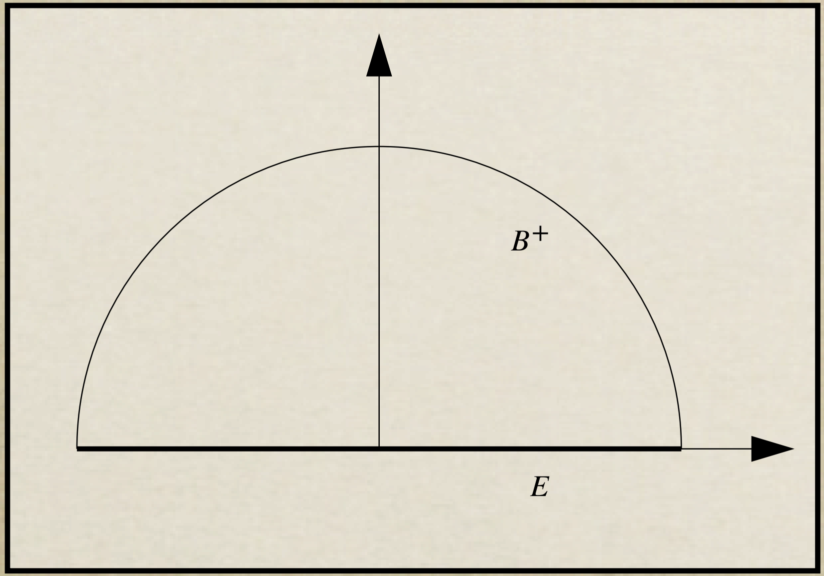

equi-partition

$$\frac{\varepsilon^3}{\delta^2}\int_{B+}|D^2u|^2\,dx + \frac{\delta^2}{\varepsilon}\int_{B+}W(u)\,dx + \lambda_\varepsilon\delta\int_E V(Tu)\,d\mathcal{H}^{N-1}$$

boundary $\delta \approx \varepsilon\lambda_\varepsilon^{-\frac{1}{3}}$

interior $\frac{\delta^2}{\varepsilon} \approx \varepsilon\lambda_\varepsilon^{-\frac{2}{3}} \to 0$

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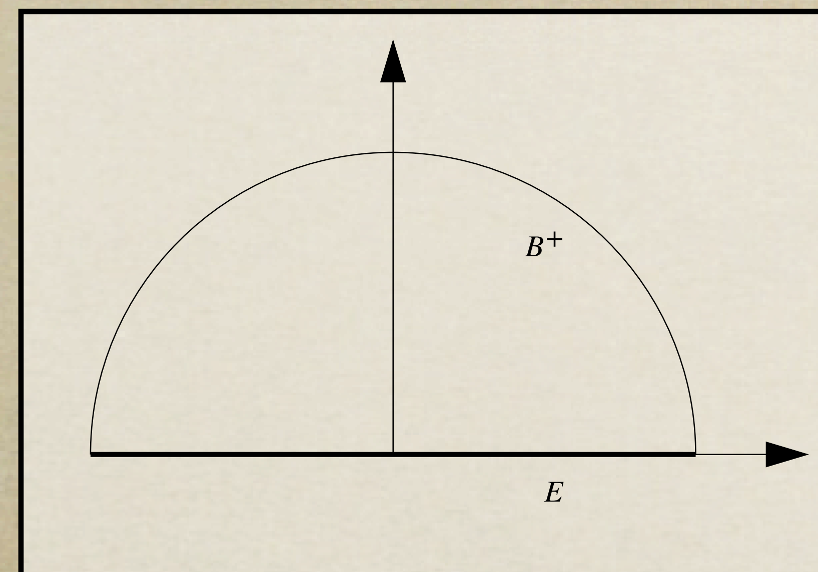

equi-partition

$$\frac{\varepsilon^3}{\delta^2}\int_{B^+}|D^2u|^2\,dx + \frac{\delta^2}{\varepsilon}\int_{B^+}W(u)\,dx + \lambda_\varepsilon\delta\int_E V(Tu)\,d\mathcal{H}^{N-1}$$

boundary $\delta \approx \varepsilon\lambda_\varepsilon^{-\frac{1}{3}}$

interior $\dfrac{\delta^2}{\varepsilon} \approx \varepsilon\lambda_\varepsilon^{-\frac{2}{3}} \to 0$

energy $\approx \varepsilon\lambda_\varepsilon^{\frac{2}{3}}$

identify regimes (λ_ε vs ε)

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$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

three regimes:

$$\varepsilon \lambda_\varepsilon^{\frac{2}{3}} \to L$$

identify regimes (λ_ε vs ε)

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$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

three regimes:

$\varepsilon \lambda_\varepsilon^{\frac{2}{3}} \to L$ → $L \in (0, \infty)$ critical

identify regimes (λ_ε vs ε)

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$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

three regimes:

$\varepsilon \lambda_\varepsilon^{\frac{2}{3}} \to L$

$L \in (0, \infty)$ critical

$L = 0$ subcritical

identify regimes (λ_ε vs ε)

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$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

three regimes:

$$\varepsilon \lambda_\varepsilon^{\frac{2}{3}} \to L$$

- $L = \infty$ supercritical
- $L \in (0, \infty)$ critical
- $L = 0$ subcritical

subcritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to 0\right)$

$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}$$

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subcritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to 0\right)$

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$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu) \, d\mathcal{H}^{N-1}$$

$$u_\varepsilon \xrightarrow{L^1} u \in BV\big(\Omega; \{a, b\}\big)$$
$$Tu_\varepsilon \xrightarrow{L^1} v \in L^1\big(\partial\Omega; \{\alpha, \beta\}\big)$$

subcritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to 0\right)$

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$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\, d\mathcal{H}^{N-1}$$

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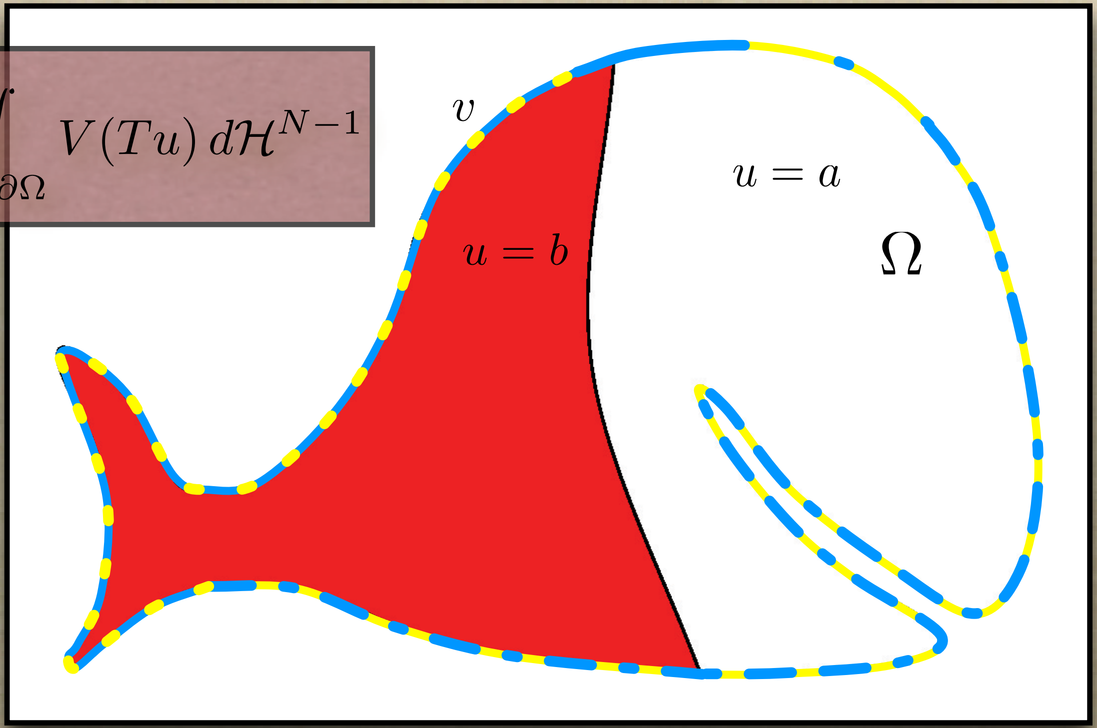

subcritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to 0\right)$

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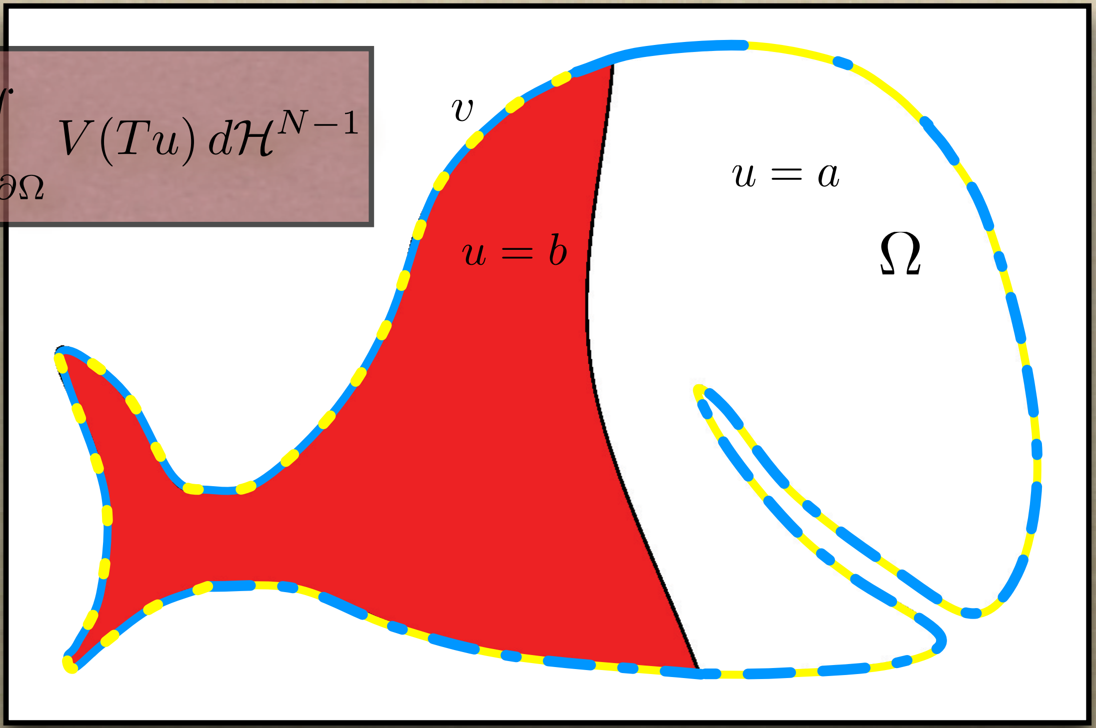
$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2 u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\, d\mathcal{H}^{N-1}$$



$$u_\varepsilon \xrightarrow{L^1} u \in BV\left(\Omega; \{a, b\}\right)$$
$$Tu_\varepsilon \xrightarrow{L^1} v \in L^1\left(\partial\Omega; \{\alpha, \beta\}\right)$$

Γ-limit

$$\mathcal{F}_\varepsilon \xrightarrow{\Gamma} m\mathrm{Per}_\Omega(\{u = a\}) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta} \sigma(z,\xi)\mathcal{H}^{N-1}(\{Tu = z\} \cap \{v = \xi\})$$

m transition between interior wells

$\sigma(z,\xi)$ transition between interior well z and boundary well ξ

supercritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to \infty\right)$

$$\mathcal{F}_\varepsilon(u) := \varepsilon^3 \int_\Omega |D^2u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}$$

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supercritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to \infty\right)$

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$$u_\varepsilon \xrightarrow{L^1} u \in BV\big(\Omega; \{a, b\}\big)$$

$$Tu_\varepsilon \xrightarrow{L^1} \text{const} = \xi \in \{\alpha, \beta\}$$

supercritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to \infty\right)$

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$$u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega; \{a,b\})$$
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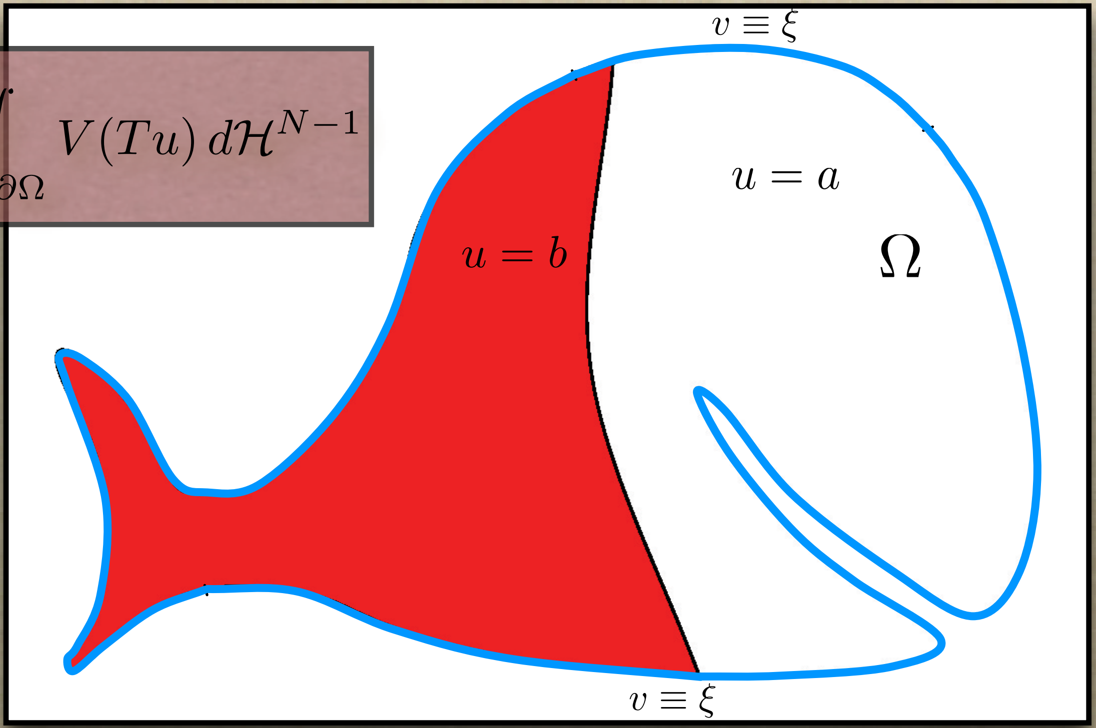

supercritical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to \infty\right)$

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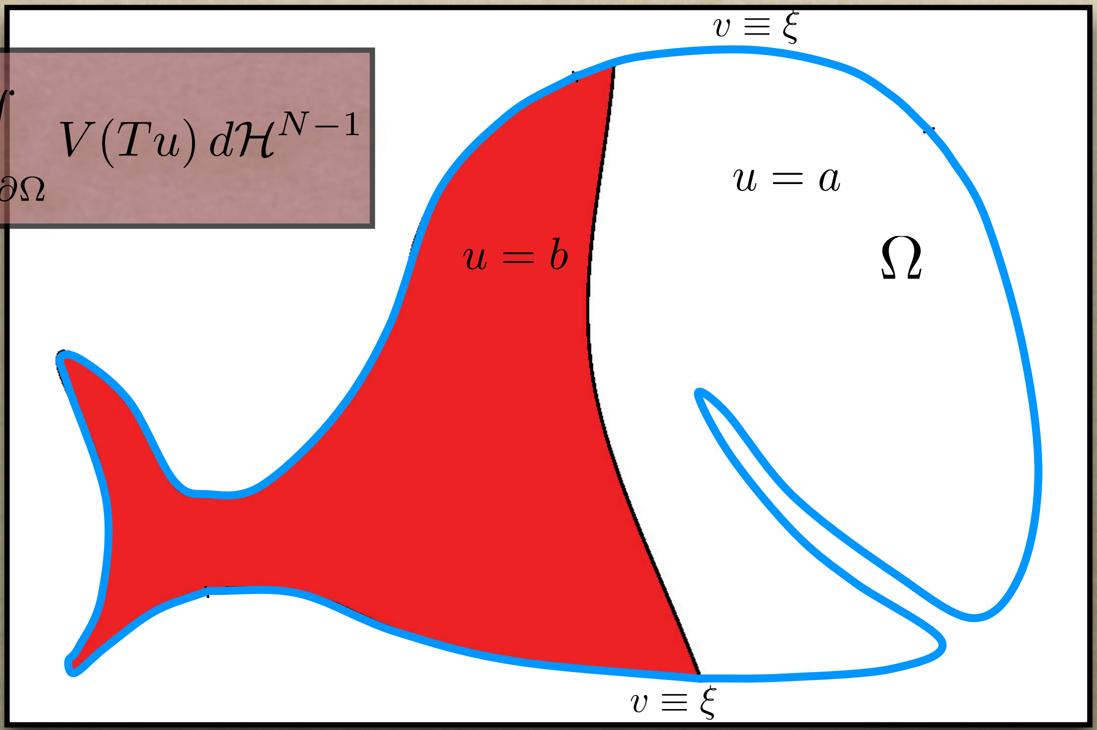

$$u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega;\{a,b\})$$
$$Tu_\varepsilon \xrightarrow{L^1} \text{const} = \xi \in \{\alpha,\beta\}$$

Γ-limit

$$\mathcal{F}_\varepsilon \xrightarrow{\Gamma} m\mathrm{Per}_\Omega(\{u=a\})$$
$$+\sigma(a,\xi)\mathcal{H}^{N-1}(\{Tu=a\}) + \sigma(b,\xi)\mathcal{H}^{N-1}(\{Tu=b\})$$

m transition between interior wells

$\sigma(z,\xi)$ transition between interior well z and boundary well ξ

critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

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critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

$$u_\varepsilon \xrightarrow{L^1} u \in BV\big(\Omega; \{a,b\}\big)$$
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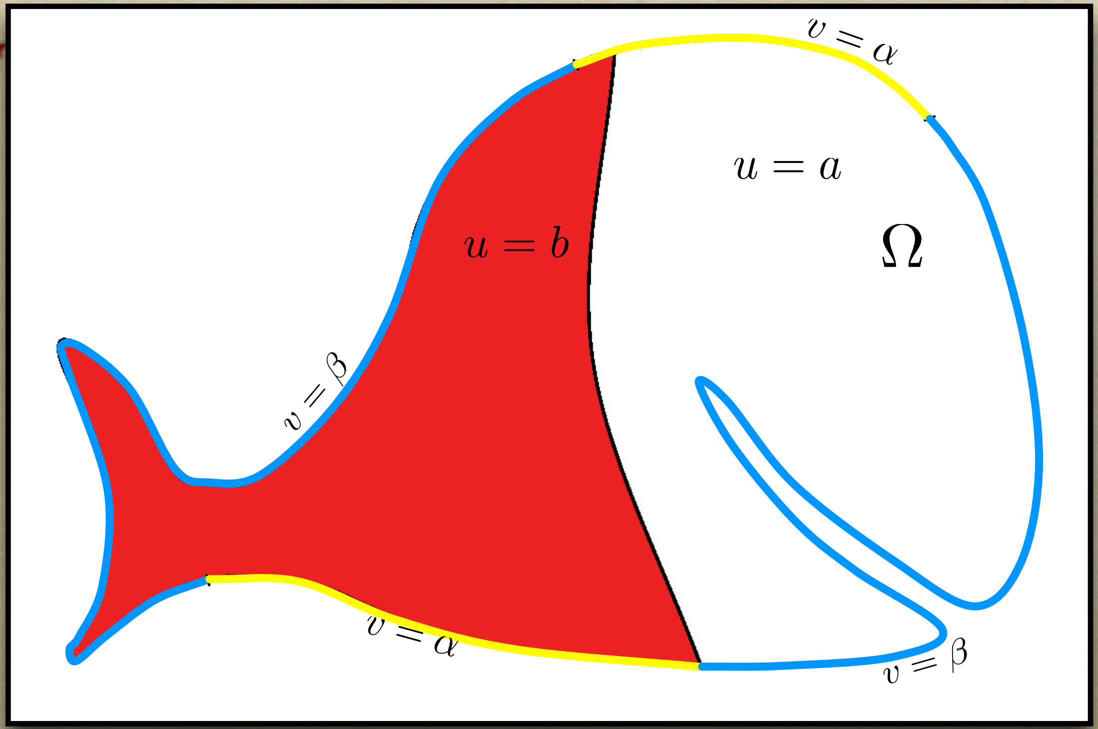

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critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

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$$u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega;\{a,b\})$$
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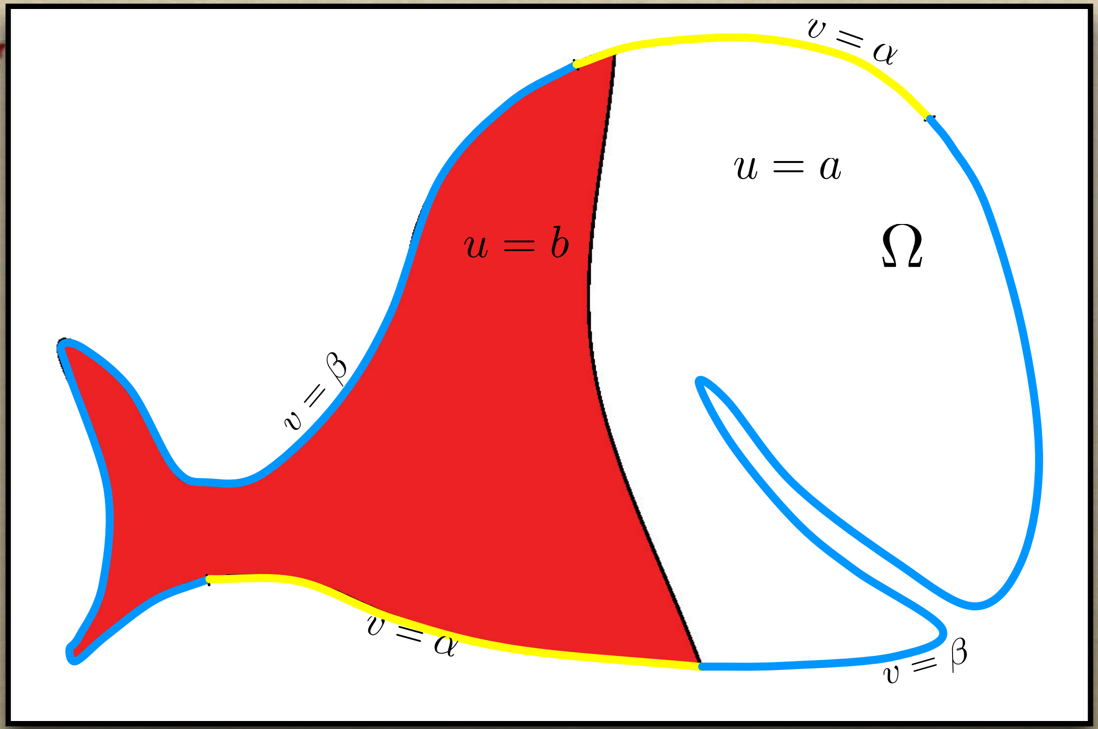

lower bound

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant m\mathrm{Per}_\Omega(\{u=a\}) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta} \sigma(z,\xi)\mathcal{H}^{N-1}(\{Tu=z\}\cap\{v=\xi\}) + \underline{c}L\mathrm{Per}_{\partial\Omega}(\{v=\alpha\})$$

m transition between interior wells

$\sigma(z,\xi)$ transition between interior well z and boundary well ξ

$\underline{c}$ lower bound estimate on the transition between boundary wells

critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

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$$u_\varepsilon \xrightarrow{L^1} u \in BV\left(\Omega;\{a,b\}\right)$$
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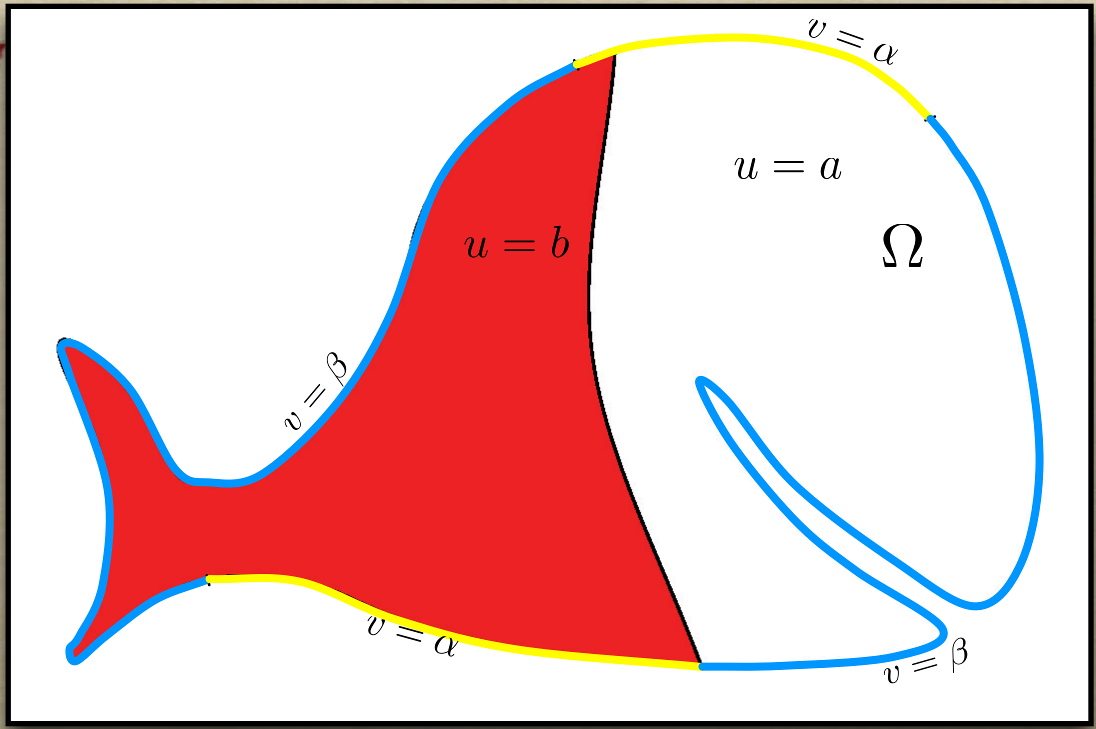

lower bound

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant m\mathrm{Per}_\Omega\left(\{u=a\}\right) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}\left(\{Tu=z\}\cap\{v=\xi\}\right) + \underline{c}L\mathrm{Per}_{\partial\Omega}\left(\{v=\alpha\}\right)$$

upper bound

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leqslant m\mathrm{Per}_\Omega\left(\{u=a\}\right) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}\left(\{Tu=z\}\cap\{v=\xi\}\right) + \overline{c}L\mathrm{Per}_{\partial\Omega}\left(\{v=\alpha\}\right)$$

critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

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$$u_\varepsilon \xrightarrow{L^1} u \in BV(\Omega;\{a,b\})$$
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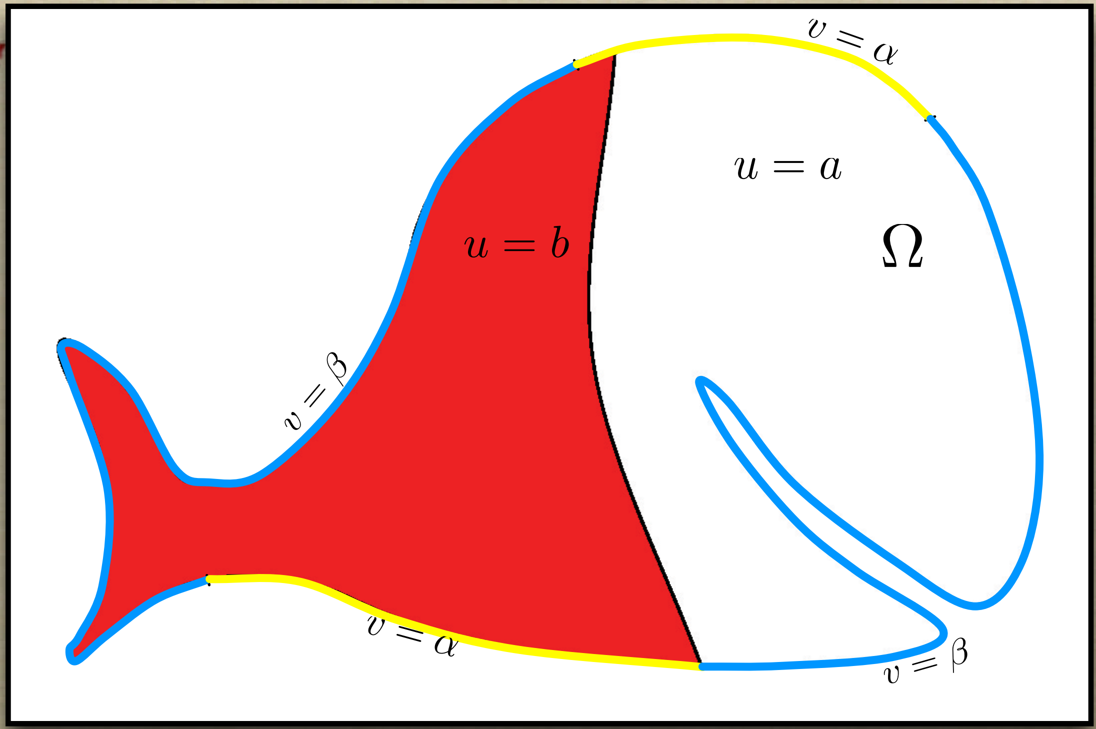

lower bound

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant m\mathrm{Per}_\Omega(\{u=a\}) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}(\{Tu=z\}\cap\{v=\xi\}) + \underline{c}L\mathrm{Per}_{\partial\Omega}(\{v=\alpha\})$$

upper bound

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critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

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$$u_\varepsilon \xrightarrow{L^1} u \in BV\big(\Omega;\{a,b\}\big)$$
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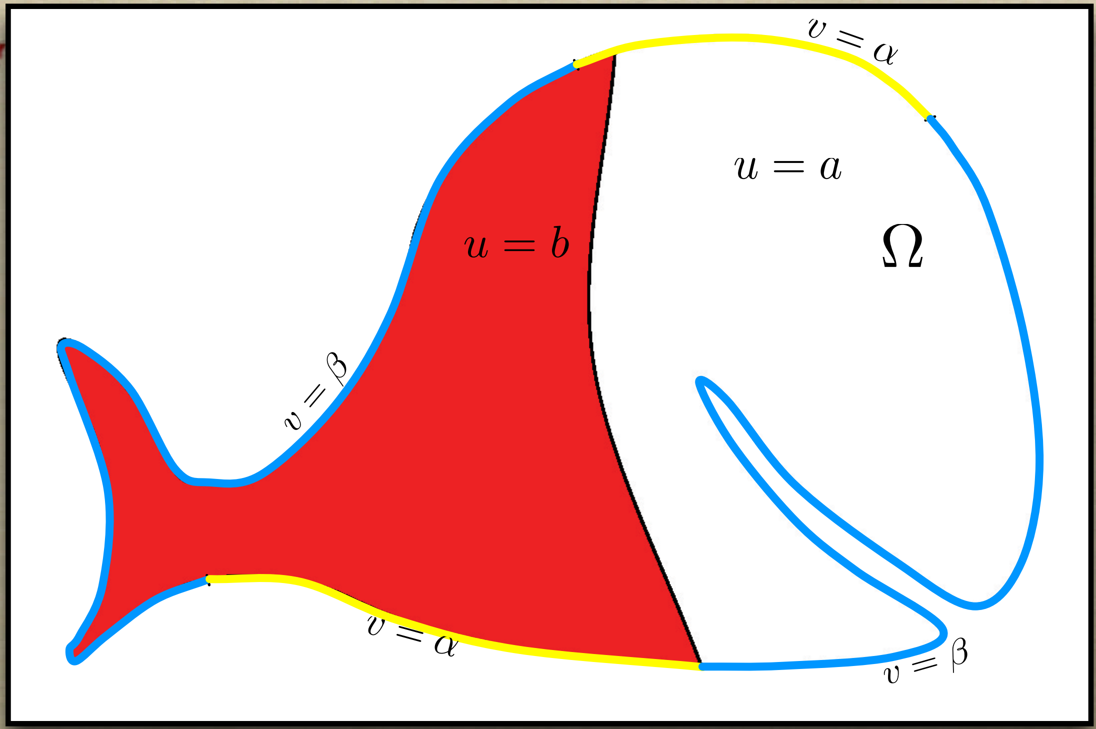
$$\sup_\varepsilon \|\nabla(Tu_\varepsilon)\|_{L^\infty(\partial\Omega)} < \infty$$



lower bound

$$\liminf_{\varepsilon\to0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant m\mathrm{Per}_\Omega\big(\{u=a\}\big) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}\big(\{Tu=z\}\cap\{v=\xi\}\big) + \underline{c}L\mathrm{Per}_{\partial\Omega}\big(\{v=\alpha\}\big)$$

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$$\limsup_{\varepsilon\to0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leqslant m\mathrm{Per}_\Omega\big(\{u=a\}\big) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}\big(\{Tu=z\}\cap\{v=\xi\}\big) + \overline{c}L\mathrm{Per}_{\partial\Omega}\big(\{v=\alpha\}\big)$$

critical case $\left(\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0,\infty)\right)$

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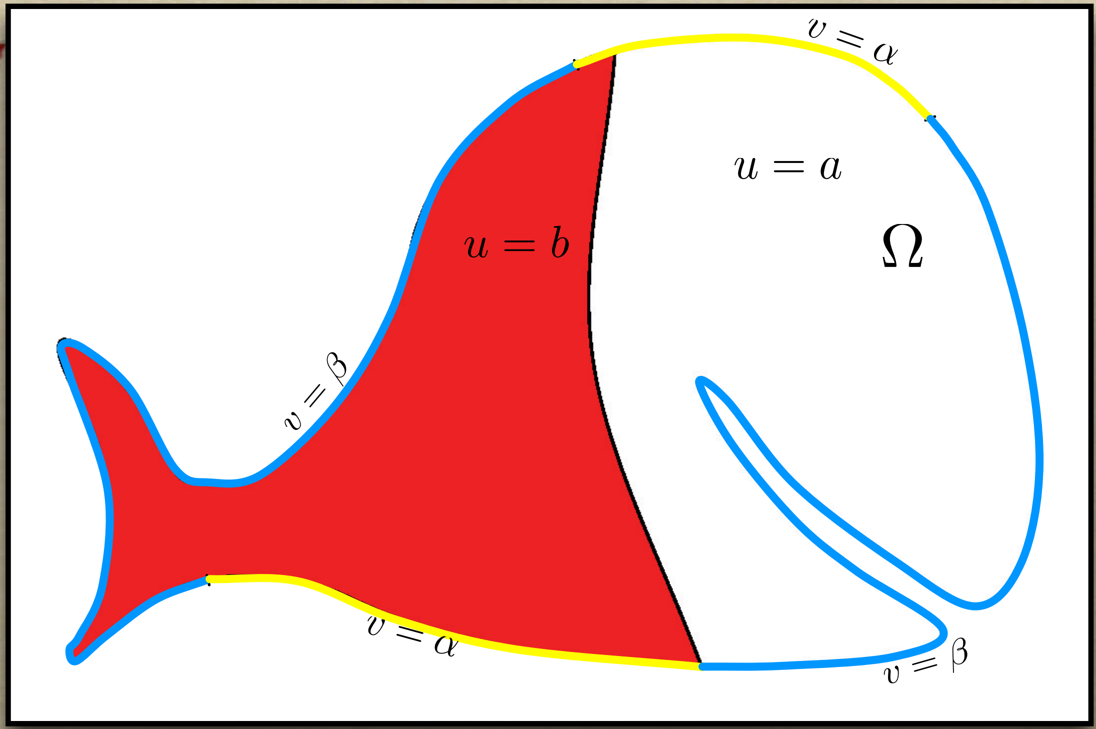

lower bound

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant m\mathrm{Per}_\Omega(\{u=a\}) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}(\{Tu=z\}\cap\{v=\xi\}) + \frac{2}{7}\bar{c}L\mathrm{Per}_{\partial\Omega}(\{v=\alpha\})$$

upper bound

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leqslant m\mathrm{Per}_\Omega(\{u=a\}) + \sum_{z=a,b}\sum_{\xi=\alpha,\beta}\sigma(z,\xi)\mathcal{H}^{N-1}(\{Tu=z\}\cap\{v=\xi\}) + \bar{c}L\mathrm{Per}_{\partial\Omega}(\{v=\alpha\})$$

remarks on the critical case

lower bound $\stackrel{?}{=}$ upper bound

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lower bound $\overset{?}{=}$ upper bound

$\Downarrow$

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$\Downarrow$

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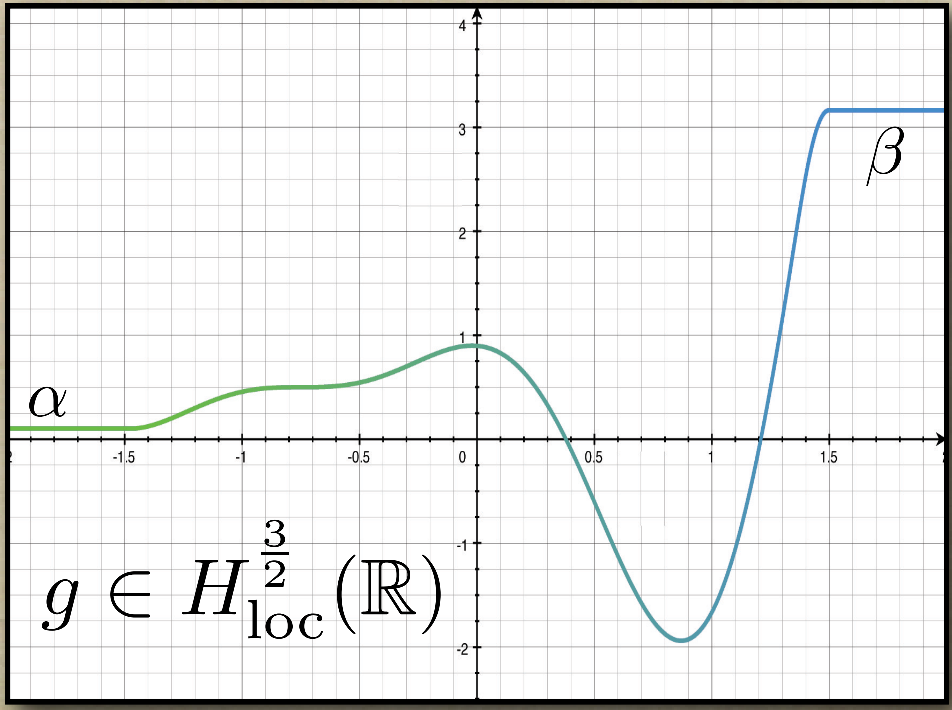

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$$\underline{c} := \inf \left\{ \frac{1}{8} \int_{-R}^{R} \int_{-R}^{R} \frac{\left|g'(x) - g'(y)\right|^2}{|x-y|^2} \, dx \, dy + \int_{-R}^{R} V\big(g(x)\big) \, dx : \right.$$
$$\left. g \in H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}), g' \in H^{\frac{1}{2}}(\mathbb{R}), g(-t) = \alpha, \, g(t) = \beta, \, \forall t \geqslant R, \, R > 0 \right\}$$

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lower bound $\overset{?}{=}$ upper bound

$$\Downarrow$$

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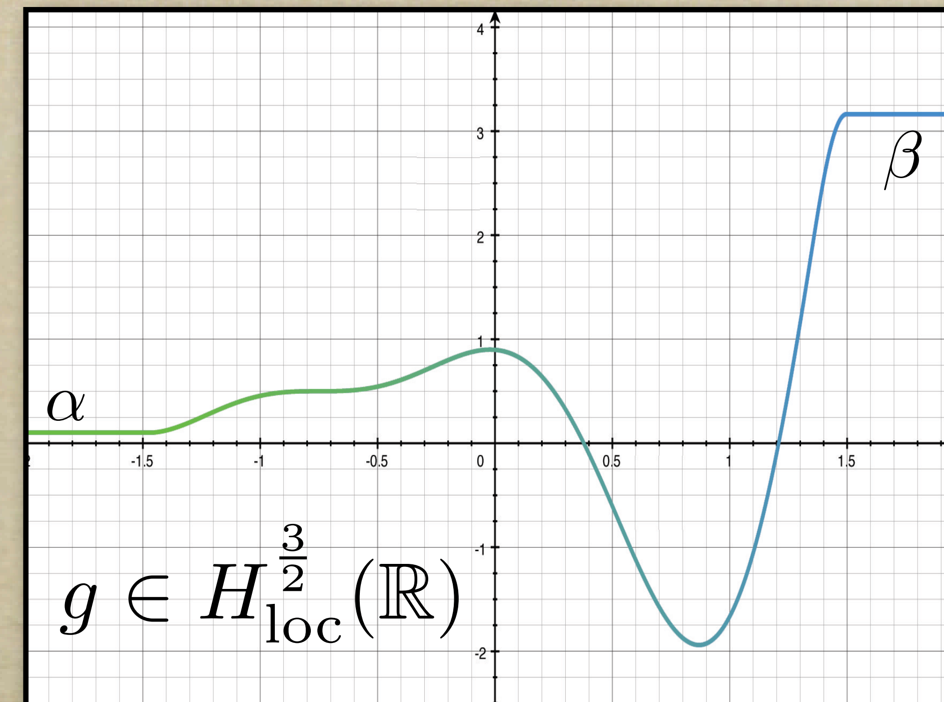

$$\underline{c} := \inf \left\{ \frac{1}{8} \int_{-R}^{R} \int_{-R}^{R} \frac{\left| g'(x) - g'(y) \right|^2}{|x-y|^2} \, dx \, dy + \int_{-R}^{R} V\big(g(x)\big) \, dx : \\ g \in H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}), g' \in H^{\frac{1}{2}}(\mathbb{R}), g(-t) = \alpha, \; g(t) = \beta, \; \forall t \geqslant R, \; R > 0 \right\}$$

$$\overline{c} := \inf \left\{ \frac{7}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\left| g'(x) - g'(y) \right|^2}{|x-y|^2} \, dx \, dy + \int_{-\infty}^{\infty} V\big(g(x)\big) \, dx : \\ g \in H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}), g(-t) = \alpha, \; g(t) = \beta, \; \forall t \geqslant R, \; R > 0 \right\}$$

remarks on the critical case

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lower bound $\overset{?}{=}$ upper bound

$\Downarrow$

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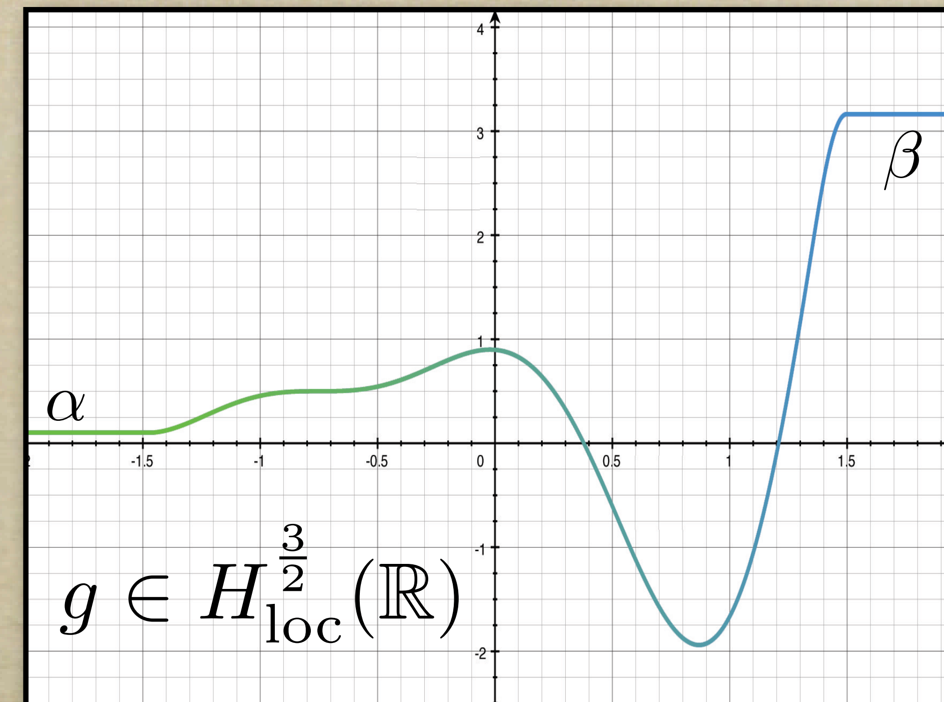


$$\underline{c} := \inf \left\{ \frac{1}{8} \int_{-R}^{R} \int_{-R}^{R} \frac{\left| g'(x) - g'(y) \right|^2}{|x-y|^2} \, dx \, dy + \int_{-R}^{R} V\big(g(x)\big) \, dx : \; g \in H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}), \boxed{g' \in H^{\frac{1}{2}}(\mathbb{R})}, g(-t) = \alpha, \; g(t) = \beta, \; \forall t \geqslant R, \; R > 0 \right\}$$

$$\overline{c} := \inf \left\{ \frac{7}{16} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\left| g'(x) - g'(y) \right|^2}{|x-y|^2} \, dx \, dy + \int_{-\infty}^{\infty} V\big(g(x)\big) \, dx : \; g \in H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}), g(-t) = \alpha, \; g(t) = \beta, \; \forall t \geqslant R, \; R > 0 \right\}$$

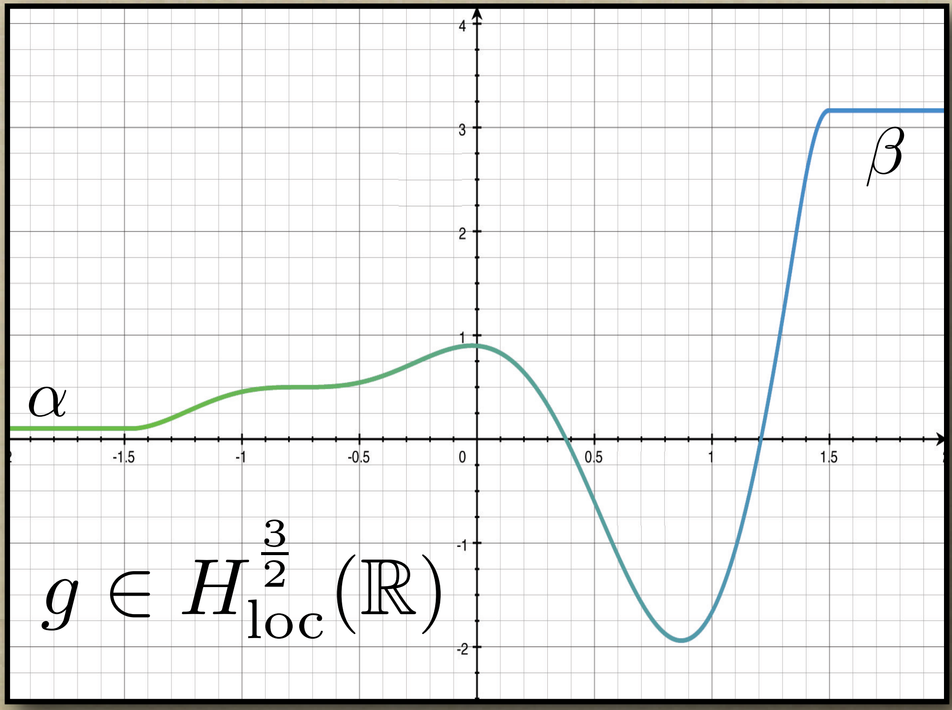
remarks on the critical case

lower bound $\overset{?}{=}$ upper bound

$\Downarrow$

Γ -limit ?

transition between boundary wells



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$$\underline{c} := \inf \left\{ \boxed{\frac{1}{8}} \int_{-R}^{R} \int_{-R}^{R} \frac{\left|g'(x) - g'(y)\right|^2}{|x-y|^2}\, dx\, dy + \int_{-R}^{R} V\left(g(x)\right) dx : \right.$$
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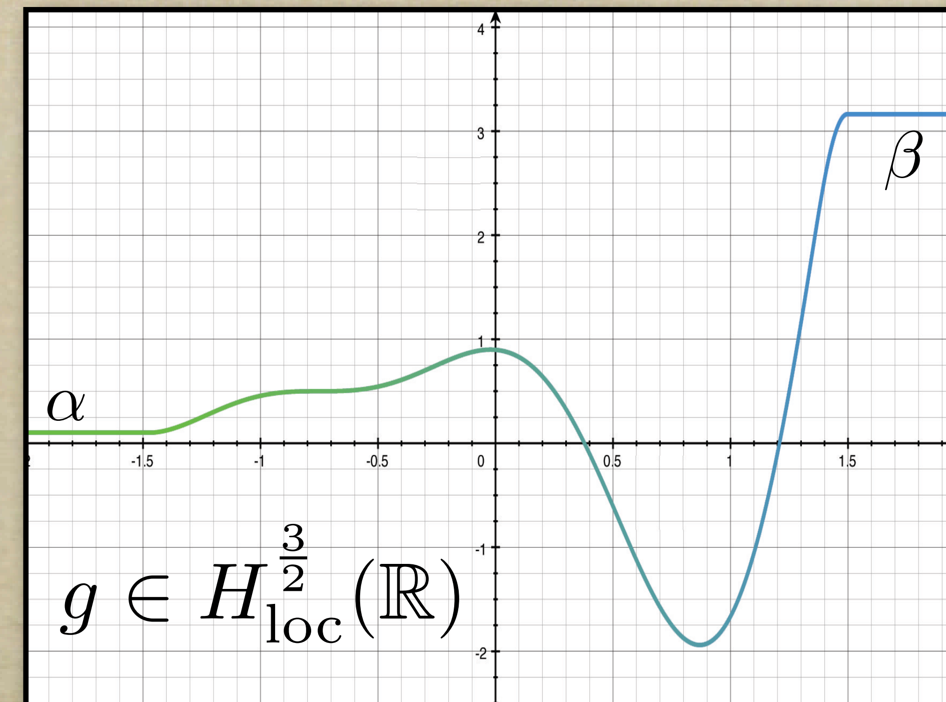
remarks on the critical case

lower bound $\overset{?}{=}$ upper bound

$\Downarrow$

Γ -limit ?

transition between boundary wells



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$$\inf_{\substack{w \in H^2_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)) \\ Tw(\cdot,0)=g}} \left\{ \frac{\iint_{T_R^+} \left|D^2 w(x,y)\right|^2 dx\,dy}{\int_0^R \int_0^R \frac{\left|g'(x)-g'(y)\right|^2}{|x-y|^2}\,dx\,dy} \right\}$$

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lower bound $\overset{?}{=}$ upper bound

⇓

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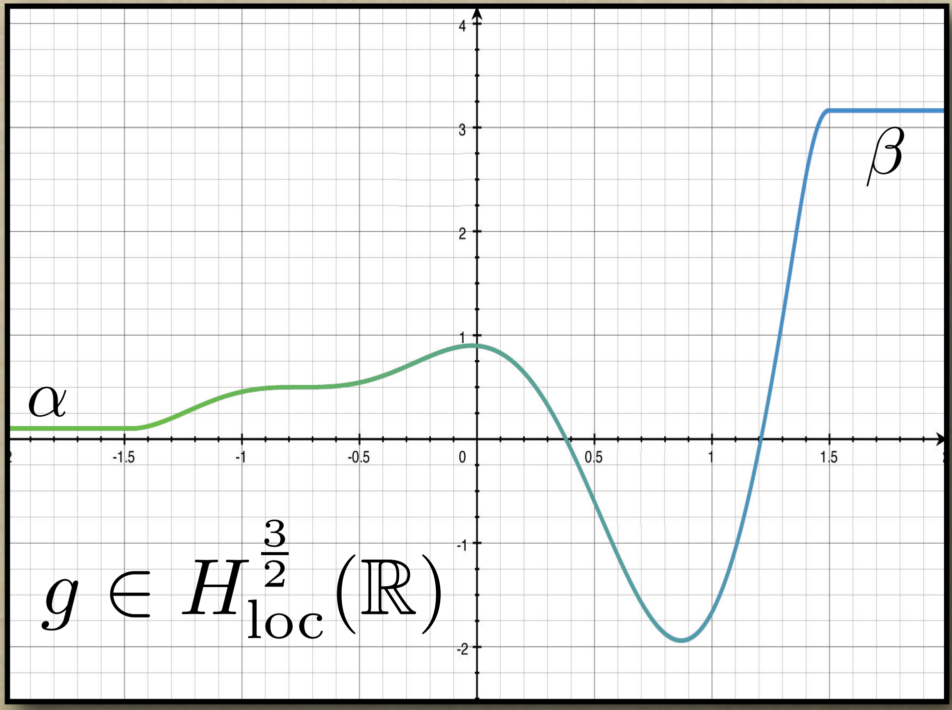

$$\frac{1}{8} \leqslant \inf_{\substack{w \in H^2_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)) \\ Tw(\cdot,0)=g}} \left\{ \frac{\iint_{T_R^+} \left|D^2 w(x,y)\right|^2 \, dx\, dy}{\int_0^R \int_0^R \frac{\left|g'(x)-g'(y)\right|^2}{|x-y|^2} \, dx\, dy} \right\}$$

fundamental theorem of calculus
+ fubini-like inequality

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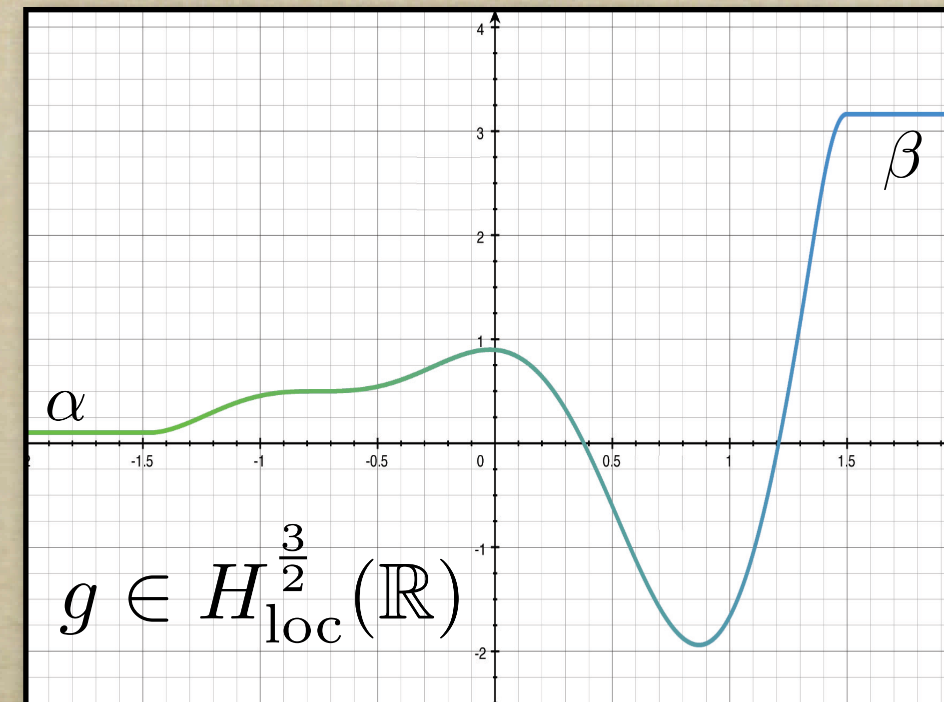
Γ-limit ?

transition between boundary wells



$$\frac{1}{8} \leqslant \inf_{\substack{w \in H^2_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)) \\ Tw(\cdot,0)=g}} \left\{ \frac{\iint_{T_R^+} \left|D^2 w(x,y)\right|^2 \, dx\, dy}{\int_0^R \int_0^R \frac{\left|g'(x)-g'(y)\right|^2}{|x-y|^2} \, dx\, dy} \right\} \leqslant \frac{7}{16}$$

fundamental theorem of calculus
+ fubini-like inequality

particular lifting

the end

historic context
problem
identify regimes
subcritical case
supercritical case
critical case
remarks

the end

historic context
problem
identify regimes
subcritical case
supercritical case
critical case
remarks

thank you

constants

back

historic context
problem
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$$m := \inf \left\{ \int_{-R}^{R} W\big(f(t)\big) + \left|f''(t)\right|^2 dt \;\; f \in H^2_{\mathrm{loc}}(\mathbb{R}), \quad f(-t)=a,\, f(t)=b,\, \forall t \geqslant R,\, R>0 \right\}$$

$$\sigma(z,\xi) := \inf \left\{ \int_{-R}^{R} W\big(f(t)\big) + \left|f''(t)\right|^2 dt \;\; f \in H^2_{\mathrm{loc}}(0,\infty), \quad f(0)=\xi,\, f(t)=z,\, \forall t \geqslant R,\, R>0 \right\}$$

$$c := \inf \left\{ \frac{1}{8} \int_{-R}^{R} \int_{-R}^{R} \frac{\left|g'(x) - g'(y)\right|^2}{|x-y|^2}\, dx\, dy + \frac{1}{4} \int_{-R}^{R} \frac{\left|g'(x)\right|^2}{R-x}\, dx + \frac{1}{4} \int_{-R}^{R} \frac{\left|g'(x)\right|^2}{R+x}\, dx + \int_{-R}^{R} V\big(g(x)\big)\, dx : g \in H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}), \quad g(-t) = \alpha,\, g(t) = \beta,\, \forall t \geqslant R,\, R > 0 \right\}$$