Research Statement

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Current Research

My current research interests are in nonlinear analysis, specifically in Calculus of Variations and Partial Differential Equations. The problems I address are motivated by issues in physical sciences and computer vision, precisely in materials science and image processing. My main goal is to further develop and study mathematical models to tackle applications and open problems from a wide range of fields.

My research work can be divided into two areas:

1. Calculus of Variations: models for phase transitions, and for thin films;
2. Image Processing.

In what follows, I will describe the research I have conducted in each of these areas and conclude with future objectives.

1. Calculus of Variations

Recently, there has been a great interest in Calculus of Variations, which arises as a consequence of current advances in materials science. Research on certain material instabilities, such as phase transitions, the appearance of microstructures, thin films, often leads to the study of minimization problems of the type

$$\min\{I(v) : v \in \mathcal{V}\},$$

where the goal is the identification of conditions on the functional I that guarantee the existence of minimizers, or problems of the type

$$\min\{I_\varepsilon(v) : v \in \mathcal{V}\},$$

where the objective is to identify and characterize the limiting problem as $\varepsilon \to 0^+$. The resolution of these problems relies heavily on certain growth, coercivity, and convexity hypotheses, which often fail to be satisfied in the context of interesting applications, thus requiring the relaxation of the energy. The classical approaches no longer suffice, and the development of new techniques in the Calculus of Variations result in a renewed importance of the field.

Phase Transitions

Within the classical theory of fluid-fluid phase transitions, given a container Ω, the fluid density u solves the "optimal design" problem

$$\min_{\int_\Omega u\,dx = V} \int_\Omega W(u)\,dx,$$

for some V with $a|\Omega| < V < b|\Omega|$, where W is the nonnegative Gibbs free energy density satisfying $\{W = 0\} = \{a, b\}$, and a, b are the two preferred fluid densities. Since there is no energy penalization for the nucleation of phases, this problem does not have uniqueness of solution. To select the physically-preferred solution, using the Van der Waals-Cahn-Hilliard theory, we add a higher-order perturbation to the energy, and this, after re-scaling, leads to

$$\min_{\int_\Omega u\,dx = V} \left[\frac{1}{\varepsilon} \int_\Omega W(u)\,dx + \varepsilon^{2k-1} \int_\Omega |D^k u|^2\,dx \right].$$

This model was thoroughly studied for the first-order perturbation $k = 1$ in [Mod87a, Ste88], and recently for the second-order perturbation $k = 2$ in [FM00].

The behavior of energy minimizers at the boundary of the domain is of great importance in the Van de Waals-Cahn-Hilliard theory for fluid-fluid phase transitions, since it describes the effect of the container walls on the configuration of the liquid. This problem, also known as the liquid-drop problem, was studied by Modica in [Mod87b], and in a different form by Alberti, Bouchitté and Seppecher in [ABS98] for the first-order perturbation model

$$\min_{\int_\Omega u dx = V} \left[\frac{1}{\varepsilon} \int_\Omega W(u)\,dx + \varepsilon \int_\Omega |Du|^2\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}(x) \right].$$

The goal of the work in [GS] is to characterize the limiting energy for the second-order perturbation model

$$\min_{\int_\Omega u dx = V} \left[\frac{1}{\varepsilon} \int_\Omega W(u)\,dx + \varepsilon^3 \int_\Omega |D^2 u|^2\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(Tu)\,d\mathcal{H}^{N-1}(x) \right],$$

where $\lambda_\varepsilon \xrightarrow{\varepsilon \to 0^+} \infty$ and V is a nonnegative, continuous double-well potential satisfying $\{V = 0\} = \{\alpha, \beta\}$. This poses new challenges since many of the techniques developed for first-order problems, *e.g.* truncation, do not carry over to the higher-order case.

We show that the boundary layer is intrinsically connected with the transition layer in the interior of the domain. Using De Giorgi's [Gio75] notion of Γ-convergence (see also [Dal93, Bra02]), we prove that the limiting behavior of the energies has three distinct regimes depending on whether the limit of $\varepsilon\lambda_\varepsilon^{\frac{2}{3}}$ is zero, positive or infinite. The critical regime, when the limit is finite and strictly positive, *i.e.* $\varepsilon\lambda_\varepsilon^{\frac{2}{3}} \to L \in (0, \infty)$, leads to a coupled problem of bulk and surface phase transitions. In this regime we are not able to prove the existence of a Γ-limit, but we show where the limiting energy is concentrated. Precisely, we prove that

(*Lower bound*) For every $u \in BV(\Omega; \{a, b\})$ and $v \in BV(\partial\Omega; \{\alpha, \beta\})$ and for every sequence $\{u_\varepsilon\} \subset H^2(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$, we have

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant m \mathrm{Per}_\Omega(E_a) + \sum_{z=a,b} \sum_{\xi=\alpha,\beta} \sigma(z,\xi) \mathcal{H}^{N-1}\big(\{Tu = z\} \cap \{v = \xi\}\big) + cL\mathrm{Per}_{\partial\Omega}(F_\alpha);$$

(*Upper bound*) For every $u \in BV(\Omega; \{a, b\})$ and $v \in BV(\partial\Omega; \{\alpha, \beta\})$, there exists a sequence $\{u_\varepsilon\} \subset H^2(\Omega)$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$, $Tu_\varepsilon \to v$ in $L^1(\partial\Omega)$, and

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leqslant m \mathrm{Per}_\Omega(E_a) + \sum_{z=a,b} \sum_{\xi=\alpha,\beta} \sigma(z,\xi) \mathcal{H}^{N-1}\big(\{Tu = z\} \cap \{v = \xi\}\big) + CL\mathrm{Per}_{\partial\Omega}(F_\alpha),$$

where $E_a := \{x \in \Omega : u(x) = a)$ and $F_\alpha := \{x \in \partial\Omega : v = \alpha\}$, m is the energy density per unit area on the transition interfaces between the interior potential wells, $\sigma(z,\xi)$ is the interaction energy on the transition interface between a bulk well z and a boundary well ξ, c is a lower bound to the energy on a transition interface between the wells of the boundary potential, and C is an upper bound to the energy on a transition interface between the wells of the boundary potential.

When the limit is infinite ($L = \infty$), the value of the underlying field on the boundary is constant,

$$\mathcal{F}_\infty(u,v) := \begin{cases} m\mathrm{Per}_\Omega(E_a) + \displaystyle\sum_{z=a,b} \sigma(z,v)\mathcal{H}^{N-1}\big(\{Tu = z\}\big) & \text{if } u \in BV\big(\Omega;\{a,b\}\big), v \in \{\alpha,\beta\}, \\ \infty & \text{otherwise.} \end{cases}$$

In the case when the limit is zero ($L = 0$), the boundary term does not contribute to the limiting energy,

$$\mathcal{F}_0(u,v) := \begin{cases} m\mathrm{Per}_\Omega(E_a) + \displaystyle\sum_{z=a,b}\sum_{\xi=\alpha,\beta} \sigma(z,\xi)\mathcal{H}^{N-1}\big(\{Tu = z\}\cap\{v=\xi\}\big) & \text{if } \begin{array}{l} u \in BV\big(\Omega;\{a,b\}\big), \\ v \in L^1\big(\partial\Omega;\{\alpha,\beta\}\big), \end{array} \\ \infty & \text{otherwise.} \end{cases}$$

Finally, we also present other common implementations of boundary conditions for this model, which can be reduced to one of the three regimes studied.

Phase Transitions for Thin Films

Thin films are thin material layers with thickness of the order of fractions of a nanometer to several micrometers. The applications for these materials range from electronic semiconductors, to ferromagnetic, and ceramic thin films. Thin film technologies are also being developed as a means of substantially reducing the cost of production: thin film modules are expected to be cheaper to manufacture owing to their reduced material, handling and capital costs. However, thin films are manufactured using new semiconductor materials, and their engineering is complicated by the fact that their physics is, in some cases, not well understood. One of the objectives of research on thin films is to be able to predict their properties through the analysis of a two-dimensional structure which, in turn, is obtained from 3-D bulk material with vanishing thickness.

In the pioneering work [CD77, CD79a, CD79b], Ciarlet and Destuynder used asymptotic expansions to deduce a two-dimensional model for elasticity from the three-dimensional one. Without any *a priori* assumptions on the strains or stresses, the authors deduced an equivalent model to the usual plate models. Acerbi, Buttazzo and Percivale in [ABP88] introduced variational methods, related to Γ-convergence, to study dimension reduction problems, and Le Dret and Raoult in [LR93] introduced Γ-convergence in the study of thin films, by proving that the quasi-minimizers of the problem

$$\inf_{u \in H^1(\Omega;\mathbb{R}^3)} \int_{\Omega_\varepsilon} W(\nabla u)\,dx,$$

where $\Omega_\varepsilon = \omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, converge to the solutions of the problem

$$\min_{\substack{u \in H^1(\Omega;\mathbb{R}^3) \\ \partial_3 u = 0}} \int_\Omega QW_0(\nabla' u)\,dx,$$

where $\nabla' u = (\partial_1 u | \partial_2 u)$, $W_0(\xi') = \min\limits_{z \in \mathbb{R}^3} W(\xi'|z)$, for $\xi' \in \mathbb{R}^{3\times 2}$, and $QW_0(\xi')$ is the quasiconvex envelope of W_0. The Γ-convergence proved to be the method of choice for problems in dimension reduction, as can be attested by a recent wealth of articles (see [BFF00, BF01, FMJ02, SZ02, BFM03, CD06, FFL07]).

In [GSM], we derive some models for solid-solid phase transitions in thin films obtained by means of Γ-convergence. The different models are determined by the asymptotic ratio between the characteristic length scale of the phase transition and the thickness of the film. Depending on the regime, we show that a separation of scales holds or not and how this later case is related to some rigidity results.

More precisely, we use Γ-convergence to study the problem

$$\min_{u\in H^2(\Omega;\mathbb{R}^3)} \left[\frac{1}{\varepsilon}\left(\frac{1}{\varepsilon^\gamma}\int_{\Omega_\varepsilon} W\left(\nabla u\right)\,dx + \varepsilon^\gamma\int_{\Omega_\varepsilon}\left|\nabla^2 u\right|^2\,dx\right)\right],$$

where $\gamma > 0, \Omega_\varepsilon = \omega\times\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)$, and W is a continuous and nonnegative double well potential, satisfying $\{W=0\}=\{A,B\}$, p-growth and p-coercivity conditions with $p>2$, and two other conditions:

(H_1) There exist positive constants ρ and C_2 such that

$$\frac{1}{C_2}\operatorname{dist}\big(\xi,\{A,B\}\big)^q \leqslant W(\xi) \leqslant C_2\operatorname{dist}\big(\xi,\{A,B\}\big)^q \;\text{ if }\; \operatorname{dist}\big(\xi,\{A,B\}\big)\leqslant\rho\,;$$

(H_2) If $A'\neq B'$, then $W(\xi_1|\xi_2|\xi_3) = W(\xi_1|-\xi_2|\xi_3)$;

If $A'=B'$, then $W(\xi'|\xi_3) = \tilde{W}\big(|\xi'-A'|,\xi_3\big)$.

For $\gamma=1$, *i.e.*, when the phase transitions are created at the same rate as the thin film, we deduce that if $\{u_\varepsilon\}\subset H^2(\Omega;\mathbb{R}^3)$ satisfies

$$\sup_{\varepsilon>0}\left[\frac{1}{\varepsilon^\gamma}\int_\Omega W(\nabla u_\varepsilon)\,dx + \varepsilon^\gamma\int_\Omega|\nabla^2 u|^2\,dx\right]<\infty,$$

then there are functions $(u,b)\in\mathcal{V}$ and a subsequence $\{u_\varepsilon\}$ (not relabeled) such that $u_\varepsilon\to u$ and $\frac{1}{\varepsilon}\partial_3 u_\varepsilon\to b$ in $L^1(\Omega;\mathbb{R}^3)$, where b is the Cosserat vector, and

$$\mathcal{V}:=\left\{(u,b)\in W^{1,\infty}(\Omega;\mathbb{R}^3)\times L^\infty(\Omega;\mathbb{R}^3) : \partial_3 u=\partial_3 b=0, (\nabla' u,b)\in BV(\Omega;\{A,B\})\right\}.$$

Moreover, we obtain a phase transition model where the energy sees that the model came from a three-dimensional one. Indeed, we prove that the Γ-limit is

$$\Gamma-\lim_{\varepsilon\to0^+} I^1_\varepsilon(u,b) = \begin{cases} K_1\operatorname{Per}_\omega(E) & \text{if } (u,b)\in\mathcal{V},\\ +\infty & \text{otherwise,}\end{cases}$$

where $E:=\{x\in\Omega : (\nabla' u,b)=A\}$, and

$$K_1 := \inf\Big\{ I^1_\varepsilon(u\,;\,Q_\nu)\;:\;\varepsilon>0\,,\; u\in H^2_{\rm loc}\big(\mathbb{R}^2\times\big(-\tfrac{1}{2},\tfrac{1}{2}\big);\mathbb{R}^3\big), u(x)=u_0(x'\cdot\nu)+\varepsilon x_3 b_0(x'\cdot\nu) \text{ near } |x'\cdot\nu|\geqslant\frac{1}{2},$$
$$u \text{ periodic of period 1 in the } \nu^\perp \text{ direction}\Big\}<+\infty.$$

The main difficulty in the proof is the fact that when $A'=B'$, ∇u remains constant, so we lose rigidity, *i.e.*, the discontinuity set is not a union of hyper-surfaces perpendicular to ν, but can have any shape.

In the case $\gamma<1$, where the thin film is created at a higher rate than the phase transitions, we derive a fully two-dimensional model that does not include its three-dimensional origin. In fact, the limiting functional is given by

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon^\gamma(u,b) = \begin{cases} K_{2\mathrm{D}}\,\mathrm{Per}_\omega(E) & \text{if } (u,b)\in\mathcal{V},\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$K_{2\mathrm{D}} := \inf\left\{ \int_{Q'_\nu} \frac{1}{\varepsilon} W(\nabla' u\,,b) + \varepsilon\big|\nabla'(\nabla' u, b)\big|^2\,dx' \,:\, \varepsilon>0, (u,b)\in H^2(\mathbb{R}^2;\mathbb{R}^3)\times H^1(\mathbb{R}^2;\mathbb{R}^3),\right.$$
$$\left. u=u_0 \text{ and } b=b_0 \text{ near } |x'\cdot\nu| = \frac{1}{2}, (u,b) \text{ periodic of period 1 in the } \nu^\perp \text{ direction}\right\} < +\infty.$$

The last regime, when the phase transition is created at a faster rate than the thin film, is more complex. Depending on the growth condition and the compatibility of the two phases, we deduce a model without phase transitions or a phase transitions model with scale separation, *i.e.*, a model equivalent to creating the phase transition first and then the thin film. In this regime, we obtain two partial results:

If $A-B=a\otimes n$ and $A'\neq B'$, then the limiting functional is

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon^\gamma(u,b) = \begin{cases} K_\gamma\,\mathrm{Per}_\omega(E) & \text{if } (u,b)\in\mathcal{V},\\ +\infty & \text{otherwise,}\end{cases}$$

where

$$K_\gamma := \inf\left\{ I_\varepsilon^\gamma(u\,;\,Q_\nu) \,:\, \varepsilon>0\,,\, u\in H^2_{\mathrm{loc}}\big(\mathbb{R}^2\times\big(-\tfrac12,\tfrac12\big);\mathbb{R}^3\big), u(x)=u_0(x)+\varepsilon x_3 b_0(x) \text{ near } |x\cdot\nu| = \frac{1}{2},\right.$$
$$\left. u \text{ periodic of period 1 in the } \nu^\perp \text{ direction}\right\}\Bigg] < +\infty.$$

If $A'=B'$ and $\gamma\geqslant p$, then we have

$$\Gamma - \lim_{\varepsilon\to 0^+} I_\varepsilon^\gamma(u,b) = \begin{cases} 0 & \text{if } (u,b)\in\overline{\mathcal{V}},\\ +\infty & \text{otherwise,}\end{cases}$$

where $\overline{\mathcal{V}} := \Big\{ (u,b)\in W^{1,\infty}(\Omega;\mathbb{R}^3)\times L^\infty(\Omega;\mathbb{R}^3) \,:\, \nabla u\equiv(A',0), b\equiv \text{constant} \in\{A_3,B_3\}\Big\}.$

2. Image Processing

Theoretical

Several methods for image processing have been developed in recent years, such as the total-variation method for denoising introduced in [ROF92]. Although algorithms have been obtained, and simulations have been carried out, the mathematical rigorous validation of many of the underlying results is still in its infancy. Meyer in [Mey01] assembled rigorous proofs for some algorithms in image processing, but left out some of the more intricate methods.

We considered the algorithm in [OBG+05], which is an improved and more complex version of the original total-variation method of [ROF92]:

- Choose $u_0=0$ and $p_0=0\in\partial J(u_0)\cap G(\Omega)$;
- Take $u_k\in BV(\Omega)$ the solution of

$$\min_{u\in BV(\Omega)} Q_k(u) := H(u,f) + J(u) - J(u_{k-1}) - \langle p_{k-1}, u - u_{k-1}\rangle\,,$$

where $p_{k-1} \in \partial J(u_{k-1}) \cap G(\Omega)$, $\partial J(u)$ is the subdifferential of J at u, and

$$G(\Omega) := \{f \in L^2(\Omega) : f = \operatorname{div}\xi, \xi \in L^\infty(\Omega;\mathbb{R}^N), \xi\cdot n = 0 \text{ on } \partial\Omega\} = \left\{f \in L^2(\Omega) : \int_\Omega f(x)\,dx = 0\right\}$$

is a subspace of $BV(\Omega)^\star$, which contains only images with zero mean, such as true noise;

- $p_k := p_{k-1} + \lambda K^\star(f - Ku_k) \in \partial J(u_k) \cap G(\Omega).$

We proved that the method is well defined, *i.e.*, there exists a minimizer $u_k \in BV(\Omega)$, and that $p_k \in \partial J(u_k) \cap G(\Omega)$.

Moreover, we proved that the algorithm yields results within a certain range of the original noise-free image for the norm of $G(\Omega)$, which is better adapted to image denoising than the one proposed in [Mey01].

Numerical

It has been shown that except for pose variation, illumination variation is the most significant factor in the appearance of faces, more significant than the inherent differences between faces. This implies that to tackle the problem of face recognition, one must first normalize the illumination condition of the faces.

The first methods assumed the knowledge of three-dimensional face models, reflective surface models, or other properties of the faces, which are often unknown. In [BK98], it was shown that the set of images of an object under varying illumination, forms a convex cone in the space of images. This algorithm produces very good results across pose and illumination, but the computational time is very high. Motivated by two widely accepted notions about human vision: it is sensitive to scene reflectance and insensitive to the illumination conditions; and human vision reacts to local changes in contrast rather than to global brightness levels; Gross and Brajovic in [GB03] suggested to obtain an image $L(x,y)$ that enhances the local contrast and then characterize the perceived image is $R(x,y)$ by

$$R(x,y) = \frac{I(x,y)}{L(x,y)},$$

where $I(x,y)$ is the original noisy image. To obtain the $L(x,y)$, the authors used a total-variation method to preprocess the image and retrieve the reflectance.

This latter method for total-variation was linear, and it has been proven that changes to a face due to illumination are highly nonlinear, and so in [GSLT] we propose to obtain $L(x,y)$as the solution of the following nonlinear total-variation-like problem

$$\min_u \left\{\int_\Omega \left[(1-\alpha(x))|\nabla u(x)| + \alpha(x)|\nabla u(x)|^2 + \frac{\lambda}{2}|u(x) - I(x)|^2\right]\,dx\right\},$$

where $\alpha(x) = \dfrac{1}{1 + k|\nabla G_\sigma \star I(x)|^2}$, and $G_\sigma(x) = \frac{1}{\sigma}e^{-\frac{|x|^2}{4\sigma^2}}$, and k, σ and λ are constants which we train to capture the nonlinearities associated with illumination normalization. We tested this algorithm with the CMU PIE image database with excellent results.

Future Research

1. Calculus of Variations

The field of Calculus of Variations is especially well adapted to deal with problems arising from issues in physical sciences. Thus, there is a world of applications that can benefit from a rigorous mathematical analysis.

Fluid-Fluid Phase Transitions

In [GS], we show that the limiting energy concentrates and we found the critical scaling, but we were not able to construct a recovery sequence that attains the Γ-limit. It is of interest to find a recovery sequence in order to prove the existence of Γ-limit. Another problem to consider is the characterization of the different constants that characterize the Γ-limit: $m, \sigma(z,\xi)$, and c.

Solid-Solid Phase Transitions

Even in the regime of linearized elasticity, there are many interesting problem within the realm of solid-solid phase transitions.

An interesting continuation of the work in [GS] is the analogous study for solid-solid phase transitions. This can be done in the context of the liquid-drop problem as in [Mod87b], *i.e.*, for the problem

$$\min_{u\in H^2(\Omega)} \left[\frac{1}{\varepsilon}\int_\Omega W(\nabla u)\,dx + \varepsilon\int_\Omega |\nabla^2 u|^2\,dx + \int_{\partial\Omega} \sigma(T\nabla u)\,d\mathcal{H}^{N-1}\right],$$

or in the context of the line-tension effect as in [ABS98], i.e., for the problem

$$\min_{u\in H^2(\Omega)} \left[\frac{1}{\varepsilon}\int_\Omega W(\nabla u)\,dx + \varepsilon\int_\Omega |\nabla^2 u|^2\,dx + \lambda_\varepsilon\int_{\partial\Omega} V(T\nabla u)\,d\mathcal{H}^{N-1}\right].$$

Another interesting problem is to study the effect of a boundary condition which is not rank-one connected with any of the wells, but it is rank-one connected with a convex combination of the wells. Precisely, consider $A,B,C\in\mathbb{R}^{3\times 3}$ such that $A-B=a\otimes n, \operatorname{rank}(C-A), \operatorname{rank}(C-B)\geqslant 2$, and there exists $\lambda\in(0,1)$ satisfying $C-(\lambda A+(1-\lambda)B)=b\otimes m$, for some $a,b\in\mathbb{R}^3$ and $n,m\in S^2$. When studying the limiting problem of

$$\min_{\substack{u\in H^2(\Omega;\mathbb{R}^3)\\ \nabla u - C\in H^1_0(\Omega;\mathbb{R}^{3\times 3})}} \left[\frac{1}{\varepsilon}\int_\Omega W(\nabla u)\,dx + \varepsilon\int_\Omega \left|\nabla^2 u\right|^2 dx\right],$$

when $\varepsilon\to 0^+$, we note that a result from [BJ87] yields contradicting conditions for the ∇u. This implies that the scaling $\frac{1}{\varepsilon}$ is over penalizing. On the other hand, if we study the limiting problem of

$$\min_{\substack{u\in H^2(\Omega;\mathbb{R}^3)\\ \nabla u - C\in H^1_0(\Omega;\mathbb{R}^{3\times 3})}} \left[\int_\Omega W(\nabla u)\,dx + \varepsilon^2\int_\Omega |\nabla^2 u|^2\,dx\right],$$

when $\varepsilon \to 0^+$, then we obtain a zero Γ-limit. It is an interesting question to find the exact scaling for this problem, which must lie between these two scalings. In fact, there is some research on this question in simpler settings, such as [KM92]. This result would enable us to then study the Γ-limit for the problem.

My work in phase transitions, and all the challenges it involved, equipped me with the tools and knowledge necessary to attempt solving these problems.

Phase Transitions for Nonlinear Elasticity

If there are many open problems in the regime of linearized elasticity, in the regime of nonlinear elasticity, even less is known. In this regime, the conditions $W(\xi) \to \infty$ as $\det \xi \to 0^+$, and allowing two orbits of wells $\{A, B\}SO(3)$ pose difficulties that have not yet been solved. It is very important to address these issues, since those are properties of the materials observed in the nature. It is my long-term goal to address these problems. A first step is to generalize the work in fluid-fluid phase transitions in [Mod87a], then study the effect of a second order perturbation (see [FM00]) to this regime, and later to generalize solid-solid phase transitions (see [CFL02, CS06a, CS06b]).

Phase Transitions for Thin Films

As for the work in phase transitions for thin films [GSM], there are several problems left to explore: we need to complete the study of the supercritical regime, where we only obtained some partial results; we need to relax the hypothesis (H_2), which we needed for technical reasons and we are convinced that it can be removed for the case $A' \neq B'$ and relaxed for the other case; we believe that in the critical case for $A' = B'$ and $A_3 \neq B_3$, if we don't assume (H_2), then the Γ-limit is non-local, *i.e.*, the constant K_1 cannot be written in integral form. Also the same work should be extended to nonlinear elasticity.

2. Image Processing

In an area as complex as computer vision, there are countless challenges and problems to tackle. There are many known problems with the current denoising algorithms, such as staircase patterns. Meyer in [Mey01] suggested a dual-norm, which would avoid the staircase effect, and is well adapted to distinguishing between texture and noise. Although computationally expensive, this norm has obvious advantages, so new methods to implement it are needed. In my opinion, Calculus of Variations, being well adapted for problems of minimization of energies, is well equipped with tools like Γ-convergence, which could make a difference in solving these problems. In fact, there is work in using Γ-convergence for edge detector algorithms (see [DMS92, ABD03, RT05]).

In [GSLT], one of our main contributions is the construction of an energy functional that enables the identification of the best parameters for a training set of images. That functional yielded good results, but there is still a need of proving the existence of minimizer for a given training set of images. Moreover, the functional can also be improved to optimize the choice of the parameters for the face recognition algorithm as well. Another part of the work that can be developed further is the face recognition algorithm itself, since in this first stage, we are using a standard nearest neighbor method.

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