Combinatorial Optimization

Problem set 7

Assigned Friday, June 26, 2015. Due Wednesday, July 1, 2015.

1. Formulate and solve an integer program for the following scenario.

> A trader of unusual objects is traveling with a caravan that begins in city A, proceeds through cities B, C, and D, in order, and ends in city E. The trader knows of some items in each city that can be purchased and later sold in other cities. The following table lists these items, their weights, their current locations, and the profit that can be gained by selling each item in later cities along the caravan route.

			Profit if sold in			
Item	Weight	In city	B	C	D	E
1	43	A	\$200	\$300	—	\$450
2	26	A	\$150	—	\$250	\$375
3	14	B		\$85	\$130	—
4	19	B		\$110	—	\$120
5	35	C			\$225	\$340
6	23	D				\$260

> The trader's camel can carry a maximum weight of 60. What items should the trader purchase, and where should the items be sold, in order to maximize profit by the end of the caravan route?

2. A *propositional formula* with the Boolean operators $\wedge$, $\vee$, and $\neg$, meaning AND, OR, and NOT, respectively, can be defined inductively as follows:

(i) A literal (i.e., a variable or its negation) is a propositional formula.

(ii) If P and Q are propositional formulas, then the conjunction $(P) \wedge (Q)$ is a propositional formula.

(iii) If P and Q are propositional formulas, then the disjunction $(P) \vee (Q)$ is a propositional formula.

(iv) If P is a propositional formula, then the negation $\neg(P)$ is a propositional formula.

Recall that a propositional formula is in *conjunctive normal form* (CNF) if it is a conjunction of disjunctions of literals. For example, the propositional formula

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3)$$

is in conjunctive normal form.

(a) Prove that any propositional formula with the Boolean operators $\wedge$, $\vee$, and $\neg$ can be rewritten as an equivalent formula, on the same set of variables, in conjunctive normal form. (*Equivalent* means that the two formulas have the same set of satisfying variable assignments.)

[Hint: Write a formula that rules out all of the non-satisfying variable assignments.]

(b) Suppose we introduce the Boolean operators $\oplus$ and $\rightarrow$, having the meanings "exclusive OR" and "implies," respectively. Show that any propositional formula with the Boolean operators $\wedge$, $\vee$, $\neg$, $\oplus$, and $\rightarrow$ can be rewritten as an equivalent formula in conjunctive normal form.

3. Formulate and solve an integer program to determine truth values for the Boolean variables x_1, x_2, x_3, x_4, and x_5 so that the propositional formula

$$(x_1 \oplus x_2) \wedge (x_3 \vee x_4 \vee x_5) \wedge \neg[x_3 \wedge (x_4 \vee x_5)] \wedge (\bar{x}_4 \vee \bar{x}_5) \\ \wedge (x_1 \vee x_4 \vee x_5) \wedge (\bar{x}_1 \vee x_3 \vee x_5) \wedge (x_2 \vee x_4 \vee x_5) \wedge (\bar{x}_2 \vee x_3 \vee x_5)$$

is satisfied, or determine that the formula is unsatisfiable.

4. Is there a maximum possible (Euclidean) distance between the optimal solution of an integer program and the optimal solution of its LP relaxation? If so, give an upper bound for this distance and prove that it is an upper bound. If not, show how to construct, given any positive real number M, an integer program whose optimal solution is at least distance M from the optimal solution of its LP relaxation.

5. Solve the following integer program using the branch-and-bound technique.

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 6 \\ & 2x_1 + 4x_2 \leq 17 \\ & x_1 \geq 0, \quad x_2 \geq 0 \\ & x_1,\ x_2 \text{ integer.} \end{array}$$

6. In the CUBIC SUBGRAPH problem, the input is a graph $G = (V, E)$, and the question to be answered is whether there exists a subgraph $H = (V', E')$ of G that is *cubic*, meaning that every vertex in the graph H has degree 3. Describe how to formulate an integer program to solve this problem, given a graph $G = (V, E)$. Justify that your formulation correctly solves the problem.