1. Consider a project consisting of the following nine activities.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Immediate prerequisites</th>
<th>Usual time (days)</th>
<th>Crash time (days)</th>
<th>Cost per day to speed up</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>—</td>
<td>2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>B</td>
<td>—</td>
<td>6</td>
<td>3</td>
<td>$180</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>4</td>
<td>2</td>
<td>$150</td>
</tr>
<tr>
<td>D</td>
<td>B</td>
<td>2</td>
<td>1</td>
<td>$200</td>
</tr>
<tr>
<td>E</td>
<td>B</td>
<td>4</td>
<td>1</td>
<td>$75</td>
</tr>
<tr>
<td>F</td>
<td>C, D</td>
<td>3</td>
<td>1</td>
<td>$250</td>
</tr>
<tr>
<td>G</td>
<td>D</td>
<td>1</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>H</td>
<td>F</td>
<td>3</td>
<td>2</td>
<td>$100</td>
</tr>
<tr>
<td>I</td>
<td>E, F, G</td>
<td>4</td>
<td>1</td>
<td>$140</td>
</tr>
</tbody>
</table>

(a) Draw a CPM network for this project.

(b) Using the usual times:
   (i) Determine the earliest and latest times for each node.
   (ii) Determine the float for each activity.
   (iii) Determine the critical path.

(c) Formulate a linear program to determine the least expensive way to reduce the length of the project by 4 days. Solve your linear program (with Maple or otherwise) and interpret your results.

(d) Formulate a linear program to determine the shortest possible completion time that can be achieved with a budget of $900. Solve your linear program and interpret your results.

**Solution.**

(a) A CPM network for this project is shown below. Solid arrows represent activities. The edge label for an activity that cannot be sped up gives the duration of the activity; the label for an activity that can be sped up gives the interval of possible durations (from the crash time to the usual time) and the per-day speedup cost. This drawing also incorporates the information from part (b): the earliest time for each node (using the usual times) is shown in green above the node, the latest time for each node is shown in red below the node, the float for each node is shown in blue below the corresponding arrow, and the critical path is highlighted in yellow.
(b) Let $u_i$ denote the usual time of activity $i$. Let $e_j$ and $l_j$ denote the earliest time and latest time of node $j$, respectively, using the usual times.

(i) The earliest time of node 0 is 0. For $j \neq 0$ the earliest time of node $j$ is found as follows: for each incoming edge (including dummy edges), add the earliest time of the edge’s start node to the duration (i.e., the usual time) of the corresponding activity; and then take the maximum of all of these sums. Note that the duration of a dummy edge is zero.

<table>
<thead>
<tr>
<th>Node</th>
<th>Earliest time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$e_0 + u_A = 0 + 2 = 2$</td>
</tr>
<tr>
<td>2</td>
<td>$e_0 + u_B = 0 + 6 = 6$</td>
</tr>
<tr>
<td>3</td>
<td>$e_1 + u_C = 2 + 4 = 6$</td>
</tr>
<tr>
<td>4</td>
<td>$e_2 + u_D = 6 + 2 = 8$</td>
</tr>
<tr>
<td>5</td>
<td>$e_2 + u_E = 6 + 4 = 10$</td>
</tr>
<tr>
<td>6</td>
<td>$\max{e_3 + 0, e_4 + 0} = \max{6,8} = 8$</td>
</tr>
<tr>
<td>7</td>
<td>$e_6 + u_F = 8 + 3 = 11$</td>
</tr>
<tr>
<td>8</td>
<td>$e_4 + u_G = 8 + 1 = 9$</td>
</tr>
<tr>
<td>9</td>
<td>$e_7 + u_H = 11 + 3 = 14$</td>
</tr>
<tr>
<td>10</td>
<td>$\max{e_5 + 0, e_7 + 0, e_8 + 0} = \max{10,11,9} = 11$</td>
</tr>
<tr>
<td>11</td>
<td>$e_{10} + u_1 = 11 + 4 = 15$</td>
</tr>
<tr>
<td>12</td>
<td>$\max{e_9 + 0, e_{11} + 0} = \max{14,15} = 15$</td>
</tr>
</tbody>
</table>

The latest time of the last node is equal to its earliest time. For each other node, the latest time is found as follows: for each outgoing edge (including dummy edges), subtract the duration of the corresponding activity from the latest time of the edge’s end node; and then take the minimum of all of these differences.

<table>
<thead>
<tr>
<th>Node</th>
<th>Latest time</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$e_{12} = 15$</td>
</tr>
<tr>
<td>11</td>
<td>$l_{12} - 0 = 15$</td>
</tr>
<tr>
<td>10</td>
<td>$l_{11} - u_4 = 15 - 4 = 11$</td>
</tr>
<tr>
<td>9</td>
<td>$l_{12} - 0 = 15$</td>
</tr>
<tr>
<td>8</td>
<td>$l_{10} - 0 = 11$</td>
</tr>
<tr>
<td>7</td>
<td>$\min{l_9 - u_H, l_{10} - 0} = \min{15 - 3, 11} = 11$</td>
</tr>
<tr>
<td>6</td>
<td>$l_{11} - u_F = 11 - 3 = 8$</td>
</tr>
<tr>
<td>5</td>
<td>$l_{10} - 0 = 11$</td>
</tr>
<tr>
<td>4</td>
<td>$\min{l_6 - 0, l_8 - u_G} = \min{8,11 - 1} = 8$</td>
</tr>
<tr>
<td>3</td>
<td>$l_6 - 0 = 8$</td>
</tr>
<tr>
<td>2</td>
<td>$\min{l_4 - u_D, l_5 - u_E} = \min{8 - 2,11 - 4} = 6$</td>
</tr>
<tr>
<td>1</td>
<td>$l_3 - u_C = 8 - 4 = 4$</td>
</tr>
<tr>
<td>0</td>
<td>$\min{l_1 - u_A, l_2 - u_B} = \min{4 - 2,6 - 6} = 0$</td>
</tr>
</tbody>
</table>

(ii) The float of an activity is equal to the latest time of its end node, minus the earliest time of its start node, minus its duration.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Float</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$l_1 - e_0 - u_A = 4 - 0 - 2 = 2$</td>
</tr>
<tr>
<td>B</td>
<td>$l_2 - e_0 - u_B = 6 - 0 - 6 = 0$</td>
</tr>
<tr>
<td>C</td>
<td>$l_3 - e_1 - u_C = 8 - 2 - 4 = 2$</td>
</tr>
<tr>
<td>D</td>
<td>$l_4 - e_2 - u_D = 8 - 6 - 2 = 0$</td>
</tr>
<tr>
<td>E</td>
<td>$l_5 - e_2 - u_E = 11 - 6 - 4 = 1$</td>
</tr>
<tr>
<td>F</td>
<td>$l_7 - e_6 - u_F = 11 - 8 - 3 = 0$</td>
</tr>
<tr>
<td>G</td>
<td>$l_8 - e_4 - u_G = 11 - 8 - 1 = 2$</td>
</tr>
<tr>
<td>H</td>
<td>$l_9 - e_7 - u_H = 15 - 11 - 3 = 1$</td>
</tr>
<tr>
<td>I</td>
<td>$l_{11} - e_{10} - u_1 = 15 - 11 - 4 = 0$</td>
</tr>
</tbody>
</table>
For completeness, we can also compute the floats for the dummy activities, even though they do not represent actual activities in the project:

<table>
<thead>
<tr>
<th>Dummy activity</th>
<th>Float</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 → 6</td>
<td>(t_6 - e_3 - 0 = 8 - 6 - 0 = 2)</td>
</tr>
<tr>
<td>4 → 6</td>
<td>(t_6 - e_4 - 0 = 8 - 8 - 0 = 0)</td>
</tr>
<tr>
<td>5 → 10</td>
<td>(t_{10} - e_5 - 0 = 11 - 10 - 0 = 1)</td>
</tr>
<tr>
<td>7 → 10</td>
<td>(t_{10} - e_7 - 0 = 11 - 11 - 0 = 0)</td>
</tr>
<tr>
<td>8 → 10</td>
<td>(t_{10} - e_8 - 0 = 11 - 9 - 0 = 2)</td>
</tr>
<tr>
<td>9 → 12</td>
<td>(t_{12} - e_9 - 0 = 15 - 14 - 0 = 1)</td>
</tr>
<tr>
<td>11 → 12</td>
<td>(t_{12} - e_{11} - 0 = 15 - 15 - 0 = 0)</td>
</tr>
</tbody>
</table>

(iii) A critical path is a path from the beginning of the project (node 0) to the completion of the project (node 12) consisting entirely of critical activities. There is a unique critical path in this project, consisting of the activities B—D—F—I.

(c, d) First we define the variables and their domains. For \(i \in \{B, C, D, E, F, H, I\}\), let \(d_i\) denote the duration of activity \(i\) (possibly sped up from its usual time). For \(0 \leq j \leq 12\), let \(t_j\) denote the time at which node \(j\) occurs. All of these variables are nonnegative.

Next we list the network constraints, because we will need the network constraints in any linear program for this project, and these constraints come from the CPM network we drew in part (a). Each activity gives us a sequence constraint:

\[
\begin{align*}
  t_1 - t_0 & \geq 2 \\
  t_2 - t_0 & \geq d_B \\
  t_3 - t_1 & \geq d_C \\
  t_4 - t_2 & \geq d_D \\
  t_5 - t_2 & \geq d_E \\
  t_7 - t_6 & \geq d_F \\
  t_8 - t_4 & \geq 1 \\
  t_9 - t_7 & \geq d_H \\
  t_11 - t_{10} & \geq 1 \\
  t_12 - t_{11} & \geq 0 \\
  t_{12} - t_{10} & \geq d_I 
\end{align*}
\]

Each dummy edge gives us a dummy constraint:

\[
\begin{align*}
  t_6 - t_3 & \geq 0 \\
  t_{10} - t_5 & \geq 0 \\
  t_{12} - t_9 & \geq 0 \\
  t_6 - t_4 & \geq 0 \\
  t_{10} - t_7 & \geq 0 \\
  t_{12} - t_{11} & \geq 0 \\
  t_{10} - t_8 & \geq 0 
\end{align*}
\]

Each activity that can be sped up gives us a pair of duration constraints:

\[
\begin{align*}
  3 \leq d_B & \leq 6 \\
  2 \leq d_C & \leq 4 \\
  1 \leq d_D & \leq 2 \\
  1 \leq d_E & \leq 4 \\
  2 \leq d_H & \leq 3 \\
  1 \leq d_I & \leq 4 
\end{align*}
\]

Now, our objective in part (c) is to minimize the total speedup cost, which is

\[
180(6-d_B)+150(4-d_C)+200(2-d_D)+75(4-d_E)+250(3-d_F)+100(3-d_H)+140(4-d_I).
\]

The usual project completion time is 15 days, so reducing the length of the project by 4 days means completing it in 11 days, which gives us the deadline constraint \(t_{12} \leq 11\). Therefore our linear program for part (c) is

\[
\begin{align*}
\text{minimize} & \quad \text{[total speedup cost]} \\
\text{subject to} & \quad \text{[all network constraints]} \\
& \quad t_{12} \leq 11 \\
& \quad \text{all variables nonnegative.}
\end{align*}
\]
Our objective in part (d) is to minimize the project completion time, which is $t_{12}$, subject to the budget constraint $[\text{total speedup cost}] \leq 900$. So our linear program for part (d) is

minimize $t_{12}$
subject to
[all network constraints]
[total speedup cost] $\leq 900$
all variables nonnegative.

The following Maple worksheet solves both of these linear programs. (It is convenient to solve them together because they share almost all of their constraints.)

```
> restart;
> with(Optimization);

[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve,
QPSolve]

> sequence:=[t1-t0>=2,t2-t0>=dB,t3-t1>=dC,t4-t2>=dD,t5-t2>=dE,
t7-t6=t0,t8-t4>=dB,t9-t7>=dH,t10-t11>=dI];

sequence := [2 \leq t_1 - t_0, dB \leq t_2 - t_0, dC \leq t_3 - t_1, dD \leq t_4 - t_2,
dE \leq t_5 - t_2, dF \leq t_7 - t_6, 1 \leq t_8 - t_4, dH \leq t_9 - t_7, dI \leq t_{10} - t_{11}]

> dummy:=[t6-t3>=0,t6-t4>=0,t10-t5>=0,t10-t7>=0,t10-t8>=0,
t12-t9>=0,t12-t11>=0];

dummy := [0 \leq t_6 - t_3, 0 \leq t_6 - t_4, 0 \leq t_{10} - t_5, 0 \leq t_{10} - t_7, 0 \leq t_{10} - t_8,
0 \leq t_{12} - t_9, 0 \leq t_{12} - t_{11}]

> duration:=[3<=dB,dB<=6,2<=dC,dC<=4,1<=dD,dD<=2,1<=dE,dE<=4,
1<=dF,dF<=3,2<=dH,dH<=3,1<=dI,dI<=4];

duration := [3 \leq d_B, d_B \leq 6, 2 \leq d_C, d_C \leq 4, 1 \leq d_D, d_D \leq 2, 1 \leq d_E, d_E \leq 4,
1 \leq d_F, d_F \leq 3, 2 \leq d_H, d_H \leq 3, 1 \leq d_I, d_I \leq 4]

> network:=[op(sequence),op(dummy),op(duration)];

network := [2 \leq t_1 - t_0, dB \leq t_2 - t_0, dC \leq t_3 - t_1, dD \leq t_4 - t_2, dE \leq t_5 - t_2,
dF \leq t_7 - t_6, 1 \leq t_8 - t_4, dH \leq t_9 - t_7, dI \leq t_{11} - t_{10}, 0 \leq t_6 - t_3,
0 \leq t_6 - t_4, 0 \leq t_{10} - t_5, 0 \leq t_{10} - t_7, 0 \leq t_{10} - t_8, 0 \leq t_{12} - t_9,
0 \leq t_{12} - t_{11}, 3 \leq d_B, d_B \leq 6, 2 \leq d_C, d_C \leq 4, 1 \leq d_D, d_D \leq 2,
1 \leq d_E, d_E \leq 4, 1 \leq d_F, d_F \leq 3, 2 \leq d_H, d_H \leq 3, 1 \leq d_I, d_I \leq 4]

> cost:=(dB,dC,dD,dE,dF,dH,dI)->180*(6-dB)+150*(4-dC)
+200*(2-dD)+75*(4-dE)+250*(3-dF)+100*(3-dH)+140*(4-dI);

cost := (d_B, d_C, d_D, d_E, d_F, d_H, d_I) \rightarrow 180 \ast (6-d_B)+150 \ast (4-d_C)
+200 \ast (2-d_D)+75 \ast (4-d_E)+250 \ast (3-d_F)+100 \ast (3-d_H)+140 \ast (4-d_I);

cost := (dB, dC, dD, dE, dF, dH, dI) \rightarrow 3990 - 180 dB - 150 d_C - 200 d_D
- 75 d_E - 250 d_F - 100 d_H - 140 d_I

> constraintsC:=[op(network),t12=11];

constraintsC := [2 \leq t_1 - t_0, dB \leq t_2 - t_0, dC \leq t_3 - t_1, dD \leq t_4 - t_2,
dE \leq t_5 - t_2, dF \leq t_7 - t_6, 1 \leq t_8 - t_4, dH \leq t_9 - t_7, dI \leq t_{11} - t_{10},
0 \leq t_6 - t_3, 0 \leq t_6 - t_4, 0 \leq t_{10} - t_5, 0 \leq t_{10} - t_7, 0 \leq t_{10} - t_8,
0 \leq t_{12} - t_9, 0 \leq t_{12} - t_{11}, 3 \leq d_B, d_B \leq 6, 2 \leq d_C, d_C \leq 4, 1 \leq d_D,
dD \leq 2, 1 \leq d_E, d_E \leq 4, 1 \leq d_F, d_F \leq 3, 2 \leq d_H, d_H \leq 3, 1 \leq d_I,
dI \leq 4, t_{12} \leq 11]
The optimal activity durations in part (c) are \( d_B = 4, d_C = 4, d_D = 2, d_E = 4, d_F = 3, d_I = 2, \) and \( d_t = 2 \). This means that the least expensive way to reduce the length of the project by 4 days is to speed up activity B by 2 days from its usual time, activity H by 1 day from its usual time, and activity I by 2 days from its usual time. The optimal objective value (i.e., the minimum total speedup cost) is \( 740 \).

The optimal activity durations in part (d) are \( d_B = 4, d_C = 4, d_D = 2, d_E = 4, d_F = 2.36, d_H = 2, \) and \( d_t = 2 \), with an optimal objective value of 10.36 days. This means that the best way to apply the budget of \$900\) in order to reduce the project completion time is to speed up activity B by 2 days from its usual time, activity F by 0.64 day from its usual time, activity H by 1 day from its usual time, and activity I by 2 days from its usual time. □
2. Rural Residence, Inc. (RRI) manufactures and builds prefab log homes. The logs are cut at their plant and delivered to the site. All other materials, such as roofing, doors, windows, etc., are purchased from other companies. The tasks involved in building one of their homes are shown in the table below.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Immediate prerequisites</th>
<th>Usual time (days)</th>
<th>Crash time (days)</th>
<th>Cost per day to speed up</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Prepare site</td>
<td>—</td>
<td>2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>B. Adjust design to site</td>
<td>A</td>
<td>2</td>
<td>1</td>
<td>$300</td>
</tr>
<tr>
<td>C. Cut logs for house</td>
<td>A</td>
<td>3</td>
<td>2</td>
<td>$250</td>
</tr>
<tr>
<td>D. Obtain other materials</td>
<td>B</td>
<td>7</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>E. Excavate basement</td>
<td>B</td>
<td>2</td>
<td>1</td>
<td>$700</td>
</tr>
<tr>
<td>F. Pour foundation</td>
<td>E</td>
<td>3</td>
<td>2</td>
<td>$350</td>
</tr>
<tr>
<td>G. Ship logs</td>
<td>C</td>
<td>5</td>
<td>3</td>
<td>$125</td>
</tr>
<tr>
<td>H. Assemble logs</td>
<td>F, G</td>
<td>8</td>
<td>5</td>
<td>$150</td>
</tr>
<tr>
<td>I. Complete roof, doors, etc.</td>
<td>D, H</td>
<td>5</td>
<td>4</td>
<td>$250</td>
</tr>
<tr>
<td>J. Prepare for utilities</td>
<td>D, H</td>
<td>5</td>
<td>3</td>
<td>$300</td>
</tr>
<tr>
<td>K. Connect utilities</td>
<td>J</td>
<td>2</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>L. Finish interior</td>
<td>J</td>
<td>7</td>
<td>4</td>
<td>$200</td>
</tr>
<tr>
<td>M. Landscape lot</td>
<td>I, K</td>
<td>2</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

In addition to the tasks above, RRI must maintain a trailer and security guard at the job site from the time that the excavation of the basement begins until the interior is finished, at a cost of $210 per day.

Determine a project schedule that will minimize the total cost of the project.

**Solution.** A CPM network for this project is shown below. Note that we can reuse node 10 as the start node for both activities I and J, because they have the same prerequisites. The excavation of the basement (activity E) begins at node 2, and the finishing of the interior (activity L) ends at node 14, so the trailer and security guard must be maintained between the two times represented by those nodes.

We will formulate a linear program for this problem, based on the CPM network. For \( i \in \{B, C, E, F, G, H, I, J, L\} \), let \( d_i \) denote the duration of activity \( i \) (possibly sped up from its usual time). For \( 0 \leq j \leq 17 \), let \( t_j \) denote the time at which node \( j \) occurs. All of these variables are nonnegative.

The network constraints come from the CPM network. Each activity gives us a sequence constraint:

\[
\begin{align*}
  t_1 - t_0 & \geq 2 \\
  t_2 - t_1 & \geq d_B \\
  t_3 - t_1 & \geq d_C \\
  t_4 - t_2 & \geq 7 \\
  t_5 - t_2 & \geq d_E \\
  t_6 - t_5 & \geq d_F \\
  t_7 - t_3 & \geq d_G \\
  t_9 - t_8 & \geq d_H \\
  t_{11} - t_{10} & \geq d_I \\
  t_{12} - t_{10} & \geq d_J \\
  t_{13} - t_{12} & \geq 2 \\
  t_{14} - t_{12} & \geq d_L \\
  t_{16} - t_{15} & \geq 2
\end{align*}
\]
Each dummy edge gives us a dummy constraint:

\[
\begin{align*}
    t_8 - t_6 & \geq 0 \\
    t_8 - t_7 & \geq 0 \\
    t_{10} - t_4 & \geq 0
\end{align*}
\]

Each activity that can be sped up gives us a pair of duration constraints:

\[
\begin{align*}
    1 \leq d_B & \leq 2 \\
    2 \leq d_C & \leq 3 \\
    1 \leq d_E & \leq 2 \\
    2 \leq d_F & \leq 3 \\
    3 \leq d_G & \leq 5 \\
    3 \leq d_I & \leq 5 \\
    4 \leq d_J & \leq 5 \\
    4 \leq d_L & \leq 7
\end{align*}
\]

Our objective is to minimize the total project cost. The portion of the project cost under our control is the speedup cost, plus the cost of maintaining the trailer and security guard from the time of node 2 until the time of node 14 at $210 per day. Therefore, the quantity we aim to minimize is

\[
300(2 - d_B) + 250(3 - d_C) + 700(2 - d_E) + 350(3 - d_F) + 125(5 - d_G) \\
+ 150(8 - d_H) + 250(5 - d_I) + 300(5 - d_J) + 200(7 - d_L) + 210(t_{14} - t_2).
\]

Our linear program is

\[
\begin{align*}
    \text{minimize} & \quad [\text{cost}] \\
    \text{subject to} & \quad [\text{all network constraints}] \\
    & \quad \text{all variables nonnegative.}
\end{align*}
\]

The following Maple worksheet solves this linear program.

\[
> \text{restart;} \\
> \text{with(Optimization);}
\]

\[
[\text{ImportMPS, Interactive, LPsolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve}]
\]

\[
> \text{sequence:=[t1-t0>=2,t2-t1>=dB,t3-t1>=dC,t4-t2>=7,t5-t2>=dE,} \\
> \quad t6-t5>=dF,t7-t3>=dG,t9-t8>=dH,t11-t10>=dI,t12-t10>=dJ,} \\
> \quad t13-t12>=2,t14-t12>=dL,t16-t15>=2];
\]

\[
\text{sequence := [2 \leq t1 - t0, dB \leq t2 - t1, dC \leq t3 - t1, 7 \leq t4 - t2, dE \leq t5 - t2,} \\
\quad dF \leq t6 - t5, dG \leq t7 - t3, dH \leq t9 - t8, dI \leq t11, t10, dJ \leq t12 - t10,} \\
\quad 2 \leq t13 - t12, dL \leq t14 - t12, 2 \leq t16 - t15]
\]

\[
> \text{dummy:=[t8-t6>=0,t8-t7>=0,t10-t4>=0,t10-t9>=0,t15-t11>=0,} \\
> \quad t15-t13>=0,t17-t14>=0,t17-t16>=0];
\]

\[
\text{dummy := [0 \leq t8 - t6, 0 \leq t8 - t7, 0 \leq t10 - t4, 0 \leq t10 - t9, 0 \leq t15 - t11,} \\
\quad 0 \leq t15 - t13, 0 \leq t17 - t14, 0 \leq t17 - t16]
\]

\[
> \text{duration:=[1<=dB,dB<=2,2<=dC,dC<=3,1<=dE,dE<=2,2<=dF,dF<=3,3<=dG,} \\
> \quad dG<=5,5<=dH,dH<=8,4<=dI,dI<=5,3<=dJ,dJ<=5,4<=dL,dL<=7];
\]

\[
\text{duration := [1 \leq dB, dB \leq 2, 2 \leq dC, dC \leq 3, 1 \leq dE, dE \leq 2, 2 \leq dF, dF \leq 3,} \\
\quad 3 \leq dG, dG \leq 5, 5 \leq dH, dH \leq 8, 4 \leq dI, dI \leq 5, 3 \leq dJ, dJ \leq 5, 4 \leq dL,} \\
\quad dL \leq 7]
\]
3. The ABC Co. manufactures its product in two plants, A and B, and sells its product in four markets, W, X, Y, and Z. The capacity in Plant A is 300 units, and in Plant B it is 350 units. The demands and per-unit shipping costs for the four markets are shown below.

<table>
<thead>
<tr>
<th>Markets</th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand</td>
<td>155</td>
<td>230</td>
<td>225</td>
<td>160</td>
</tr>
<tr>
<td>Per-unit shipping cost from A:</td>
<td>$10</td>
<td>$20</td>
<td>$15</td>
<td>$25</td>
</tr>
<tr>
<td>Per-unit shipping cost from B:</td>
<td>$5</td>
<td>$15</td>
<td>$10</td>
<td>$20</td>
</tr>
</tbody>
</table>

The usual per-unit labor cost is $95 in either plant. The other costs per unit are $50 in Plant A and $70 in Plant B. Overtime labor can be hired only at Plant A at a per-unit cost of $140. If the capacity is not adequate to meet demand, additional items can be manufactured at Plant A using overtime labor.

(a) Formulate a linear program to determine how ABC should schedule its production to meet all demand while minimizing its total costs. Solve your LP (with Maple or otherwise) and interpret the results.
(b) Solve this problem using the specialized transportation algorithm described in class. Compare this result to your result from part (a).

**Solution.** We can view this as an instance of the transportation problem. We have three origins: usual production in Plant A, usual production in Plant B, and overtime production in Plant A. We shall label these origins A, B, and O, respectively. We have four destinations: the markets W, X, Y, and Z.

The total demand is $155 + 230 + 225 + 160 = 770$ units, but the total supply at origins A and B (that is, the total capacity of both plants) is only 650 units. Therefore, we will need to produce 120 units using overtime labor, so the supply at origin O is 120.

The total per-unit production costs at each origin are $145$ at A, $165$ at B, and $190$ at O. Combining these with the shipping costs, we get the per-unit costs from each source to each destination shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$155$</td>
<td>$165$</td>
<td>$160$</td>
<td>$170$</td>
</tr>
<tr>
<td>B</td>
<td>$170$</td>
<td>$180$</td>
<td>$175$</td>
<td>$185$</td>
</tr>
<tr>
<td>O</td>
<td>$200$</td>
<td>$210$</td>
<td>$205$</td>
<td>$215$</td>
</tr>
</tbody>
</table>

(a) For $i \in \{A, B, O\}$ and $j \in \{W, X, Y, Z\}$, let $x_{ij}$ denote the number of units to be produced at origin $i$ and shipped to market $j$. All of these variables are nonnegative.

Our objective is to minimize total cost, subject to the constraints that the supply at each origin is exhausted and the demand at each destination is satisfied. Thus we have the following linear program.

\[
\begin{align*}
\text{maximize} & \quad 155x_{AW} + 165x_{AX} + 160x_{AY} + 170x_{AZ} \\
& \quad + 170x_{BW} + 180x_{BX} + 175x_{BY} + 185x_{BZ} \\
& \quad + 200x_{OW} + 210x_{OX} + 205x_{OY} + 215x_{OZ} \\
\text{subject to} & \quad x_{AW} + x_{AX} + x_{AY} + x_{AZ} = 300 \\
& \quad x_{BW} + x_{BX} + x_{BY} + x_{BZ} = 350 \\
& \quad x_{OW} + x_{OX} + x_{OY} + x_{OZ} = 120 \\
& \quad x_{AW} + x_{BW} + x_{OW} = 155 \\
& \quad x_{AX} + x_{BX} + x_{OX} = 230 \\
& \quad x_{AY} + x_{BY} + x_{OY} = 225 \\
& \quad x_{AZ} + x_{BZ} + x_{OZ} = 160 \\
& \quad \text{all variables nonnegative.}
\end{align*}
\]

The following Maple worksheet solves this linear program.

```maple
> restart;
> with(Optimization);

[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]

> cost:=(AW,AX,AY,AZ,BW,BX,BY,BZ,OW,OX,OY,OZ)->155*AW+165*AX +160*AY+170*AZ+170*BW+180*BX+175*BY+185*BZ+200*OW+210*OX +205*OY+215*OZ;

cost := (AW, AX, AY, AZ, BW, BX, BY, BZ, OW, OX, OY, OZ) ->
155 AW + 165 AX + 160 AY + 170 AZ + 170 BW + 180 BX + 175 BY + 185 BZ + 200 OW + 210 OX + 205 OY + 215 OZ
```
> supply := \[\text{AW} + \text{AX} + \text{AY} + \text{AZ} = 300, \text{BW} + \text{BX} + \text{BY} + \text{BZ} = 350, \text{OW} + \text{OX} + \text{OY} + \text{OZ} = 120\];

supply := \[\text{AW} + \text{AX} + \text{AY} + \text{AZ} = 300, \text{BW} + \text{BX} + \text{BY} + \text{BZ} = 350, \text{OW} + \text{OX} + \text{OY} + \text{OZ} = 120\]

> demand := \[\text{AW} + \text{BW} + \text{OW} = 155, \text{AX} + \text{BX} + \text{OX} = 230, \text{AY} + \text{BY} + \text{OY} = 225, \text{AZ} + \text{BZ} + \text{OZ} = 160\];

demand := \[\text{AW} + \text{BW} + \text{OW} = 155, \text{AX} + \text{BX} + \text{OX} = 230, \text{AY} + \text{BY} + \text{OY} = 225, \text{AZ} + \text{BZ} + \text{OZ} = 160\]

> constraints := \[\text{op(supply)}, \text{op(demand)}\];

constraints := \[\text{AW} + \text{AX} + \text{AY} + \text{AZ} = 300, \text{BW} + \text{BX} + \text{BY} + \text{BZ} = 350, \text{OW} + \text{OX} + \text{OY} + \text{OZ} = 120, \text{AW} + \text{BW} + \text{OW} = 155, \text{AX} + \text{BX} + \text{OX} = 230, \text{AY} + \text{BY} + \text{OY} = 225, \text{AZ} + \text{BZ} + \text{OZ} = 160\]

> LPSolve(cost(AW, AX, AY, AZ, BW, BX, BY, BZ, OW, OX, OY, OZ), constraints, assume=nonnegative);

\[
1.35825 \times 10^5, \text{[AW = 0., AX = 110.0000000000000, AY = 30.0000000000000, AZ = 160.0000000000000, BW = 155.0000000000000, BX = 0., BY = 195., BZ = 0., OW = 0., OX = 120., OY = 0., OZ = 0.]}\]

So an optimal production and shipping plan is as shown in the table below:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>110</td>
<td>30</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>155</td>
<td>195</td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>120</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The objective value of this solution (i.e., the total cost of production and shipping) is $135,825.

(b) Shown below is the initial transportation tableau, with the initial basic feasible solution constructed using the method described in the lecture: satisfy demands one by one, left to right, using the supplies top to bottom. The values of the basic squares are shown in bold in the lower left corners. Then the values of the dual variables are calculated, starting with \(v_1 = 0\) arbitrarily, and these dual values are used to compute the test values for the nonbasic squares (shown in red in the lower right corners).

\[
\begin{array}{cccc}
A: & 300 & & & & & & \\
 & 155 & 165 & 160 & 170 & 0 & 0 & \\
B: & 350 & & & & & & \\
 & 0 & 85 & 225 & 175 & 40 & 185 & \\
O: & 120 & & & & & & \\
 & 200 & 210 & 205 & 215 & 120 & & \\
\end{array}
\]

\[
\begin{array}{cccc}
A: & 300 & & & & & & \\
 & 155 & 165 & 160 & 170 & 0 & 0 & \\
B: & 350 & & & & & & \\
 & 0 & 85 & 225 & 175 & 40 & 185 & \\
O: & 120 & & & & & & \\
 & 200 & 210 & 205 & 215 & 120 & & \\
\end{array}
\]

Since all the test values are nonnegative, this solution is already optimal. So we have another optimal production and shipping plan:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>155</td>
<td>145</td>
<td>&amp;</td>
<td>&amp;</td>
</tr>
<tr>
<td>B</td>
<td>85</td>
<td>225</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>&amp;</td>
<td>&amp;</td>
<td>120</td>
<td></td>
</tr>
</tbody>
</table>

The objective value of this solution is \(155(\$155) + 145(\$165) + 85(\$180) + 225(\$175) + 40(\$185) + 120(\$215) = \$135,825\), which matches the objective value of the optimal solution found by Maple. (In fact, as it turns out, all feasible solutions are optimal for this particular instance.) 

\(\square\)
4. A manufacturer can ordinarily produce 300 units of a certain product each month and needs to schedule production for a three-month period in which the orders exceed this capacity. Inventory at the beginning of the first month is 120 units, and the demands for the successive three months are 420, 360, and 450 units. Monthly production capacity can be increased by up to 100 units at an additional cost of $8 per unit. Holding costs to manufacture in one month and ship during a later month are $2 per unit per month.

Determine a production schedule that will minimize the total cost of exceeding the usual monthly capacity and holding costs.

**Solution.** We can formulate this as an instance of the transportation problem. There are seven origins: with labels and their supplies, they are the starting inventory (I, 120), first month usual production (A, 300), first month increased production (AA, 100), second month usual production (B, 300), second month increased production (BB, 100), third month usual production (C, 300), and third month increased production (CC, 100). Three destinations are explicitly stated in the problem: with labels and their demands, they are the first month (X, 420), second month (Y, 360), and third month (Z, 450). However, the total supply is 120 + 300 + 100 + 300 + 100 + 300 + 100 = 1320, while the total demand is only 420 + 360 + 450 = 1230, so we will add a fictitious fourth destination, “unused” (U, 90), to make total supply and total demand equal.

The per-unit costs from each origin to each destination are shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$0</td>
<td>$2</td>
<td>$4</td>
<td>$6</td>
</tr>
<tr>
<td>A</td>
<td>$0</td>
<td>$2</td>
<td>$4</td>
<td>$0</td>
</tr>
<tr>
<td>AA</td>
<td>$8</td>
<td>$10</td>
<td>$12</td>
<td>$0</td>
</tr>
<tr>
<td>B</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$2$</td>
<td>$0$</td>
</tr>
<tr>
<td>BB</td>
<td>$\infty$</td>
<td>$8$</td>
<td>$10$</td>
<td>$0$</td>
</tr>
<tr>
<td>C</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>CC</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$8$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Some of the costs in this table are given as $\infty$, because they represent impossible origin-destination combinations. For example, it is impossible to use ordinary production in the third month (C) to satisfy demand in the first month (X). Because our objective is to minimize total cost, setting the costs of these impossible combinations to $\infty$ will make them prohibitively expensive to use in an optimal solution. (If $\infty$ causes implementation difficulties because arithmetic with $\infty$ is not well defined, then a very large finite cost, such as $10,000$, could be used instead; it would achieve the same result.)

Most of the entries in the U column of this cost table are zero, because they represent origin-destination combinations for which items would not be produced in the first place. For example, if a solution calls for 10 units to be “shipped” from origin AA to destination U, then the meaning is that 10 units that could be produced using increased production in the first month will go unused—they will not be used to satisfy demand. In that case, there is no need to produce them at all, so only 90 units will be produced from AA, and those 10 units that are never produced will incur no cost. However, the IU entry in the cost table is $6, because the items from origin I (the starting inventory) have already been produced, and not using them to satisfy one of the demands will incur three months of holding costs.
The initial transportation tableau appears below.

<table>
<thead>
<tr>
<th></th>
<th>X: 420</th>
<th>Y: 360</th>
<th>Z: 450</th>
<th>U: 90</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A: 300</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AA: 100</td>
<td>60</td>
<td>40</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>B: 300</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BB: 100</td>
<td>60</td>
<td>40</td>
<td>40</td>
<td>0</td>
</tr>
<tr>
<td>C: 300</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CC: 100</td>
<td>60</td>
<td>40</td>
<td>40</td>
<td>0</td>
</tr>
</tbody>
</table>

Note that the AAX square has been made basic, with a value of 0, to complete the “stair-step” pattern from the upper left to the lower right. This is important because it gives us the correct number of basic squares and causes the basic squares to correspond to the edges of a tree in the bipartite graph whose vertices are the origins and destinations.

This tableau contains negative test values, so it is not optimal. The most negative test value occurs in the AAU square, so this will be our pivot square. The corresponding pivot circuit goes AAU—CCU—CCZ—BZ—BY—AAY—AAU, so we will add $t$ to the flows in the AAU, CCZ, and BY squares and subtract $t$ from the flows in the CCU, BZ, and AAY squares. Therefore the largest allowable value of $t$ (without causing a negative flow in the CCU, BZ, or AAY square) is $t = 40$. We adjust the flows around the pivot cycle; the flow in the BZ square becomes zero, so that square falls out of the basis. Then we construct the next tableau.

The most negative test value in this tableau occurs in the BBY square, for which the pivot circuit is BBY—AAY—AAU—CCU—CCZ—BBZ—BBY. In the pivot we will add $t$ to the flows in the BBY, AAU, and CCZ squares and subtract $t$ from the flows in the AAY, CCU, and BBZ squares, so the largest permissible value of $t$ is $t = 50$. When we adjust the flows, the flow in the CCU square becomes zero, so it falls out of the basis. The
next tableau appears below.

<table>
<thead>
<tr>
<th></th>
<th>X: 420</th>
<th>Y: 360</th>
<th>Z: 450</th>
<th>U: 90</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: 120</td>
<td>120</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>A: 300</td>
<td>300</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>AA: 100</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>B: 300</td>
<td>∞</td>
<td>300</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>BB: 100</td>
<td>∞</td>
<td>50</td>
<td>50</td>
<td>2</td>
</tr>
<tr>
<td>C: 300</td>
<td>∞</td>
<td>∞</td>
<td>300</td>
<td>0</td>
</tr>
<tr>
<td>CC: 100</td>
<td>∞</td>
<td>∞</td>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ w_1 = 0 \quad w_2 = 2 \quad w_3 = 4 \quad w_4 = -8 \]

In this tableau, all test values are nonnegative, so we have reached an optimal solution. In the first month, the manufacturer should use all of the starting inventory and usual production to meet demand and should use increased production to produce an extra 10 units. Those 10 units will be used to satisfy the demand in the second month, along with all of the usual production and 50 extra units produced using increased production in the second month. An additional 50 extra units should also be produced using increased production in the second month to satisfy the demand in the third month, together with all of the usual production and 100 extra units of increased production. The cost of this solution is $120(0) + 300(0) + 0(8) + 10(10) + 90(0) + 300(0) + 50(8) + 50(10) + 300(0) + 100(8) = 1800$.

Alternatively, we may formulate this problem as a linear program. For \( i \in \{I, A, AA, B, BB, C, CC\} \) and \( j \in \{X, Y, Z, U\} \), let \( x_{ij} \) denote the number of units to be taken from origin \( i \) and used to satisfy demand at destination \( j \). These variables are all nonnegative. Actually, we do not need the variables \( x_{BX}, x_{BBX}, x_{CX}, x_{CCX}, x_{CY}, \) and \( x_{CCY} \), because these origin-destination combinations are impossible. We also do not need the variables \( x_{AU}, x_{AAU}, x_{BU}, x_{BBU}, x_{CU}, \) and \( x_{CCU} \) if we do not require that total supply equal total demand, and we can remove that requirement by formulating the constraints as inequalities. The linear program is shown below.

\[
\begin{align*}
\text{minimize} \quad & 2x_{IY} + 4x_{IZ} + 6x_{IU} + 2x_{AY} + 4x_{AZ} + 8x_{AAX} + 10x_{AAY} + 12x_{AAX} + 2x_{BZ} + 8x_{BBY} + 10x_{BBZ} + 8x_{CCZ} \\
\text{subject to} \quad & x_{IX} + x_{IY} + x_{IZ} + x_{IU} = 120 \\
& x_{AX} + x_{AY} + x_{AZ} \leq 300 \\
& x_{AAX} + x_{AAY} + x_{AAZ} \leq 100 \\
& x_{BY} + x_{BZ} \leq 300 \\
& x_{BBY} + x_{BBZ} \leq 100 \\
& x_{CZ} \leq 300 \\
& x_{CCZ} \leq 100 \\
& x_{IX} + x_{AX} + x_{AAX} \geq 420 \\
& x_{IY} + x_{AY} + x_{AAY} + x_{BY} + x_{BBY} \geq 360 \\
& x_{IZ} + x_{AZ} + x_{AAZ} + x_{BZ} + x_{BBZ} + x_{CZ} + x_{CCZ} \geq 450 \\
& \text{all variables nonnegative.}
\end{align*}
\]
The following Maple worksheet solves this linear program.

```maple
> restart;
> with(Optimization);

[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]

> cost:=(IY,IZ,IU,AY,AZ,AAX,AAY,AAZ,BZ,BBY,BBZ,CCZ)->2*IY+4*IZ+6*IU +2*AY+4*AZ+8*AAX+10*AAY+12*AAZ+2*BZ+8*BBY+10*BBZ+8*CCZ;

```cost
```:=
```

```maple
> supply:=[IX+IY+IZ+IU=120,AX+AY+AZ<=300,AAX+AAY+AAZ<=100,
BBY+BBZ<=100,CZ<=300,CCZ<=100];

```supply```

```maple
> demand:=[IX+AX+AAX>=420,IY+AY+AAY+BY+BBY>=360,
IZ+AZ+AAZ+BZ+BBZ+CZ+CCZ>=450];

```demand```

```maple
> constraints:=[op(supply),op(demand)];

```constraints```

```maple
> LPSolve(cost(IY,IZ,IU,AY,AZ,AAX,AAY,AAZ,BZ,BBY,BBZ,CCZ),
constraints,assume=nonnegative);

```

The optimal solution found by Maple is shown in the table below.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>AA</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>BB</td>
<td>60</td>
<td>40</td>
</tr>
<tr>
<td>C</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>CC</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

This solution also has cost $1800.