Combinatorial Optimization

Problem set 2: solutions

1. Consider the following two linear programs in standard form:

$$\begin{aligned} \text{maximize} \quad & c^{\mathrm{T}}x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \text{maximize} \quad & -c^{\mathrm{T}}x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Can both of these linear programs have feasible solutions with arbitrarily large objective value? If yes, give an example; if not, prove so.

Solution. Yes. For example,

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 \\ x_1 \geq 0, \quad & x_2 \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \text{maximize} \quad & -x_1 + x_2 \\ x_1 \geq 0, \quad & x_2 \geq 0 \end{aligned}$$

[Note that the set of constraints for these linear programs is empty, so the feasible region is the entire first quadrant. If you don't like $0 \times n$ matrices, you can accomplish the same thing by taking, say, A to be a zero matrix and b to be a zero vector. Or throw in the constraint $x_1 - 2x_2 - s_1 = 0$, with $s_1 \geq 0$; both linear programs are still unbounded (exercise: justify this claim).] □

2. In class we saw an example that served as a sketch of a proof of the following theorem:

> **Theorem.** *Let x be a feasible solution to a maximizing linear program (in standard form). Then either there exists a basic feasible solution whose objective value is at least as large as that of x, or else the linear program is unbounded.*

Using the example as a guide, prove this theorem.

Solution. Let the linear program be

$$\begin{aligned} \text{maximize} \quad & c^{\mathrm{T}}x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where c and x are $n \times 1$ column vectors, A is an $m \times n$ matrix, and b is an $m \times 1$ column vector.

Let x be a feasible solution to this linear program. By reordering the components of x (and the components of c, and the columns of A), we may assume without loss of generality that the first r components of x are nonzero and the remaining $n - r$ components are zero:

$$\begin{aligned} x_i \neq 0 \quad & \text{for } 1 \leq i \leq r; \\ x_i = 0 \quad & \text{for } r + 1 \leq i \leq n. \end{aligned}$$

We proceed by induction on r.

If $r = 0$, then x is the zero vector, which certainly is a basic solution: Since x is feasible, it satisfies $Ax = b$, so b must be the zero vector. So we may choose any basis of A and the corresponding basic solution will be the zero vector. Therefore, if $r = 0$, then we are done, because x itself is a basic feasible solution. This establishes the base case.

For the inductive step, suppose that $r \geq 1$. If the first r columns of A are linearly independent, then we are done, because x itself is a basic solution: we can extend the first r columns of A to a basis, and the corresponding basic solution will be x.

Otherwise, the first r columns $\{A_1, \ldots, A_r\}$ of A are linearly dependent. Therefore there exist coefficients $h_1, \ldots, h_r$, not all zero, such that $A_1h_1 + \cdots + A_rh_r = 0$. Let h be the $n \times 1$ column vector $[h_1, \ldots, h_r, 0, \ldots, 0]^{\mathrm{T}}$, so that $Ah = 0$. Note that $h \neq 0$ because not all of the coefficients $h_1, \ldots, h_r$ are zero. Also observe that $A(x+th) = Ax + tAh = b + 0 = b$ for all $t \in \mathbb{R}$, so $x + th$ will satisfy the constraints of the linear program for any scalar t, which means that $x + th$ will be a feasible solution as long as all of its components are nonnegative.

Without loss of generality, we may assume that $c^{\mathrm{T}}h \geq 0$; if not, then take $-h$ instead of h.

We now consider two cases, depending on whether h has any negative components. In the first case, h has no negative components. If $c^{\mathrm{T}}h = 0$, then $-h$ must have at least one negative component (because $h \neq 0$), so take $-h$ instead of h and go to the second case. Otherwise $c^{\mathrm{T}}h > 0$. Since h has no negative components, we have $x + th \geq x \geq 0$ for all $t \geq 0$, so $x + th$ is feasible for all $t \geq 0$. Furthermore, the objective value of $x + th$ is $c^{\mathrm{T}}(x + th) = c^{\mathrm{T}}x + t(c^{\mathrm{T}}h)$, which can be made arbitrarily large because $c^{\mathrm{T}}h > 0$. So the linear program is unbounded.

In the second case, h has at least one negative component. Let

$$t = \min \left\{ -\frac{x_i}{h_i} : h_i < 0 \right\}.$$

Let k be a coordinate at which this minimum is attained, that is, $-x_k/h_k = t$. Since $x_i > 0$ for all $1 \leq i \leq r$, and $h_i = 0$ for all $r + 1 \leq i \leq n$, we see that $t > 0$ and $1 \leq k \leq r$. Let $y = x + th$. For $1 \leq i \leq r$, if $h_i \geq 0$ then $y_i = x_i + th_i \geq x_i \geq 0$; otherwise, by the definition of t, we have $y_i = x_i + th_i \geq x_i + (-x_i/h_i)h_i = 0$. For $r + 1 \leq i \leq n$, we have $y_i = x_i + th_i = 0 + t \cdot 0 = 0$. So y is feasible, because all of its components are nonnegative. Moreover, $y_k = x_k + th_k = x_k + (-x_k/h_k)h_k = 0$, so y has strictly fewer than r nonzero components (because $y_i = 0$ for $i = k$ and for all $r + 1 \leq i \leq n$). Finally, $c^{\mathrm{T}}y = c^{\mathrm{T}}(x + th) = c^{\mathrm{T}}x + t(c^{\mathrm{T}}h) \geq c^{\mathrm{T}}x$ because $t > 0$ and $c^{\mathrm{T}}x \geq 0$, so the objective value of y is at least as large as that of x. By the inductive hypothesis, either there exists a basic feasible solution whose objective value is at least as large as that of y (and hence at least as large as that of x), or else the linear program is unbounded. This completes the inductive step, and hence the proof. □

3. Convert the following linear program to standard form. Write the initial simplex tableau and give the initial basic feasible solution. Do a pivot to bring x_2 into the basis and give the resulting basic feasible solution.

$$\begin{array}{rl} \text{maximize} & 5x_1 + 3x_2 - 2x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 10 \\ & 4x_1 + 5x_2 \leq 20 \\ & 2x_1 - 3x_2 + 2x_3 \leq 6 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0. \end{array}$$

Solution. First we insert slack variables to convert the linear program to standard form:

$$\begin{array}{rl} \text{maximize} & 5x_1 + 3x_2 - 2x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 + s_1 = 10 \\ & 4x_1 + 5x_2 + s_2 = 20 \\ & 2x_1 - 3x_2 + 2x_3 + s_3 = 6 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \quad s_3 \geq 0. \end{array}$$

Now we can write the initial simplex tableau:

x_1	x_2	x_3	s_1	s_2	s_3	z	RHS
-5	-3	2	0	0	0	1	0
1	2	1	1	0	0	0	10
4	5	0	0	1	0	0	20
2	-3	2	0	0	1	0	6

The initial basic feasible solution is

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad s_1 = 10, \quad s_2 = 20, \quad s_3 = 6.$$

We are asked to do a pivot to bring x_2 into the basis, so we need to pivot on an entry in the x_2 column. There are two positive entries in this column, which are the candidates to be the pivot entry. The test ratio for the entry 2 in the first row of the body of the tableau is $10/2 = 5$, and the test ratio for the entry 5 in the second row is $20/5 = 4$. So we pivot on the 5, because it has the minimum test ratio. After this pivot, the tableau becomes the following:

x_1	x_2	x_3	s_1	s_2	s_3	z	RHS
$-13/5$	0	2	0	$3/5$	0	1	12
$-3/5$	0	1	1	$-2/5$	0	0	2
$4/5$	1	0	0	$1/5$	0	0	4
$22/5$	0	2	0	$3/5$	1	0	18

The corresponding basic feasible solution is

$$x_1 = 0, \quad x_2 = 4, \quad x_3 = 0, \quad s_1 = 2, \quad s_2 = 0, \quad s_3 = 18. \qquad \square$$

4. Answer yes or no and prove your answer: Can a pivot move the corresponding basic feasible solution a positive distance in $\mathbb{R}^n$ while leaving the objective value unchanged?

Solution. Yes. For example:

x	s_1	s_2	z	RHS
0	0	0	1	0
2	1	0	0	5
$\boxed{1}$	0	1	0	1

The corresponding basic feasible solution is $x = 0$, $s_1 = 5$, $s_2 = 1$, with an objective value of 0. Pivoting on the indicated entry produces the following tableau:

x	s_1	s_2	z	RHS
0	0	0	1	0
0	1	-2	0	3
1	0	1	0	1

The basic feasible solution corresponding to this tableau is $x = 1$, $s_1 = 3$, $s_2 = 0$, also having an objective value of 0. So the pivot moved the basic feasible solution a positive distance in $\mathbb{R}^n$ (i.e., the two basic solutions are not identical), but the objective value is unchanged.

In general, this will happen whenever a pivot is performed in a column having a zero in the objective row (in a non-degenerate tableau). $\square$

5. Solve the following linear program by hand, using the simplex algorithm.

$$\begin{aligned}
\text{maximize} \quad & 20x_1 + 6x_2 + 8x_3 \\
\text{subject to} \quad & 6x_1 + 2x_2 + 3x_3 \le 420 \\
& 4x_1 + 3x_2 \le 200 \\
& x_3 \le 50 \\
& x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0.
\end{aligned}$$

Solution. First we insert slack variables in order to convert the linear program to standard form:

$$\begin{aligned}
\text{maximize} \quad & 20x_1 + 6x_2 + 8x_3 \\
\text{subject to} \quad & 6x_1 + 2x_2 + 3x_3 + s_1 = 420 \\
& 4x_1 + 3x_2 + s_2 = 200 \\
& x_3 + s_3 = 50 \\
& x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad s_1 \ge 0, \quad s_2 \ge 0, \quad s_3 \ge 0.
\end{aligned}$$

Now we write the initial simplex tableau:

x_1	x_2	x_3	s_1	s_2	s_3	z	RHS
-20	-6	-8	0	0	0	1	0
6	2	3	1	0	0	0	420
4	3	0	0	1	0	0	200
0	0	1	0	0	1	0	50

We will pivot in the x_1 column, because it has the most negative entry in the objective row. The test ratio for the entry 6 in that column is $420/6 = 70$, the test ratio for the entry 4 in that column is $200/4 = 50$, and we cannot pivot on the entry 0. So we pivot on the entry 4, because it has the minimum test ratio. After that pivot, we obtain the following tableau:

x_1	x_2	x_3	s_1	s_2	s_3	z	RHS
0	9	-8	0	5	0	1	1000
0	$-5/2$	3	1	$-3/2$	0	0	120
1	$3/4$	0	0	$1/4$	0	0	50
0	0	1	0	0	1	0	50

Now we pivot in the x_3 column, because it is the only column with a negative entry in the objective row. The test ratio for the entry 3 in that column is $120/3 = 40$, we cannot pivot on the entry 0, and the test ratio for the entry 1 in that column is $50/1 = 50$. So we pivot on the entry 3, because it has the minimum test ratio. After that pivot, we get this tableau:

x_1	x_2	x_3	s_1	s_2	s_3	z	RHS
0	$7/3$	0	$8/3$	1	0	1	1320
0	$-5/6$	1	$1/3$	$-1/2$	0	0	40
1	$3/4$	0	0	$1/4$	0	0	50
0	$5/6$	0	$-1/3$	$1/2$	1	0	10

Now there are no negative entries in the objective row, so this tableau is optimal. The optimal solution to the linear program is $x_1 = 50$, $x_2 = 0$, $x_3 = 40$, $s_1 = 0$, $s_2 = 0$, $s_3 = 10$, having the optimal objective value 1320. □

6. Consider the following simplex tableau (for the maximizing simplex algorithm).

x_1	x_2	x_3	s_1	s_2	s_3	s_4	z	RHS
a_1	0	0	0	40	0	25	1	700
a_2	0	0	1	1/2	0	-2	0	84
a_3	1	0	0	-2	0	5/2	0	225
a_4	0	1	0	3/2	0	1/2	0	125
a_5	0	0	0	$-5/2$	1	$-3/2$	0	65

For each part below, describe conditions on the entries $a_1, \ldots, a_5$ in the first column so that the tableau satisfies the stated condition. Try to make your answers as general as possible.

(a) The corresponding basic feasible solution is optimal and unique.

(b) The corresponding basic feasible solution is optimal but not unique, and $x_1 = 13$ in the alternative optimal basic feasible solution.

(c) The corresponding basic feasible solution is not optimal, and in the next basic feasible solution (that is, the basic feasible solution corresponding to the tableau after the next pivot in the simplex algorithm), $s_1 = 0$ and the value of the objective function is 742.

(d) The corresponding basic feasible solution is not optimal, and in the next basic feasible solution $x_3 = 0$ and $s_3 = 40$.

Solution.

(a) A non-degenerate simplex tableau represents a unique optimal bfs if and only if all nonbasic columns have positive entries in the objective row. For this to be true of this tableau, the x_1 column must be a nonbasic column with a positive entry in the objective row; so $a_1 > 0$. There are no conditions on the entries $a_2, \ldots, a_5$.

(b) A non-degenerate simplex tableau represents a non-unique optimal bfs if and only if all entries in the objective row are nonnegative and at least one nonbasic column has a zero in the objective row. So we need $a_1 = 0$. From such a tableau, an alternative optimal bfs can be obtained by pivoting in that nonbasic column on a positive entry having the minimum test ratio. In order to make $x_1 = 13$ in the alternative optimal bfs, we need that minimum test ratio to be 13. So we additionally need $a_2 \leq 84/13$, $a_3 \leq 225/13$, $a_4 \leq 125/13$, and $a_5 \leq 5$ with at least one of these inequalities satisfied by equality.

(c) In order for the corresponding bfs to be non-optimal, we need a negative entry in the objective row, so $a_1 < 0$. In order to make $s_1 = 0$ in the next bfs, we need one of the following to be true:

- We can pivot on a_2, thereby making $s_1 = 0$ because it will fall out of the basis. To make the value of the objective function be 742 in the next bfs, we need $a_2 = -2a_1$, and to ensure that a_2 has the minimum test ratio, we need $a_3 \leq 225a_2/84$, $a_4 \leq 125a_2/84$, and $a_5 \leq 65a_2/84$.
- We can pivot on a_3. To make the value of the objective function be 742 in the next bfs, we need $a_3 = -225a_1/42$. To make $s_1 = 0$ in the next bfs, we need $a_2 = 84a_3/225$. To ensure that a_3 has the minimum test ratio, we need $a_4 \leq 125a_3/225$ and $a_5 \leq 65a_3/225$.
- We can pivot on a_4. To make the value of the objective function be 742 in the next bfs, we need $a_4 = -125a_1/42$. To make $s_1 = 0$ in the next bfs, we need $a_2 = 84a_4/125$. To ensure that a_4 has the minimum test ratio, we need $a_3 \leq 225a_4/125$ and $a_5 \leq 65a_4/125$.
- We can pivot on a_5. To make the value of the objective function be 742 in the next bfs, we need $a_5 = -65a_1/42$. To make $s_1 = 0$ in the next bfs, we need $a_2 = 84a_5/65$. To ensure that a_5 has the minimum test ratio, we need $a_3 \leq 225a_5/65$ and $a_4 \leq 125a_5/65$.

(d) In order for the corresponding bfs to be non-optimal, we need a negative entry in the objective row, so $a_1 < 0$. In order to make $x_3 = 0$ in the next bfs, we need one of the following to be true:

- We can pivot on a_4, thereby making $x_3 = 0$ because it will fall out of the basis. To make $s_3 = 40$ in the next bfs, we need $a_5 = 25a_4/125$. To ensure that a_4 has the minimum test ratio, we need $a_2 \le 84a_4/125$ and $a_3 \le 225a_4/125$.
- We can pivot on a_2. To make $x_3 = 0$ in the next bfs, we need $a_4 = 125a_2/84$. To make $s_3 = 40$ in the next bfs, we need $a_5 = 25a_2/84$. To ensure that a_2 has the minimum test ratio, we need $a_3 \le 225a_2/84$.
- We can pivot on a_3. To make $x_3 = 0$ in the next bfs, we need $a_4 = 125a_3/225$. To make $s_3 = 40$ in the next bfs, we need $a_5 = 25a_3/225$. To ensure that a_3 has the minimum test ratio, we need $a_2 \le 84a_3/225$.

Note that we cannot pivot on a_5, because that would cause s_3 to fall out of the basis, which would make it impossible to have $s_3 = 40$. □