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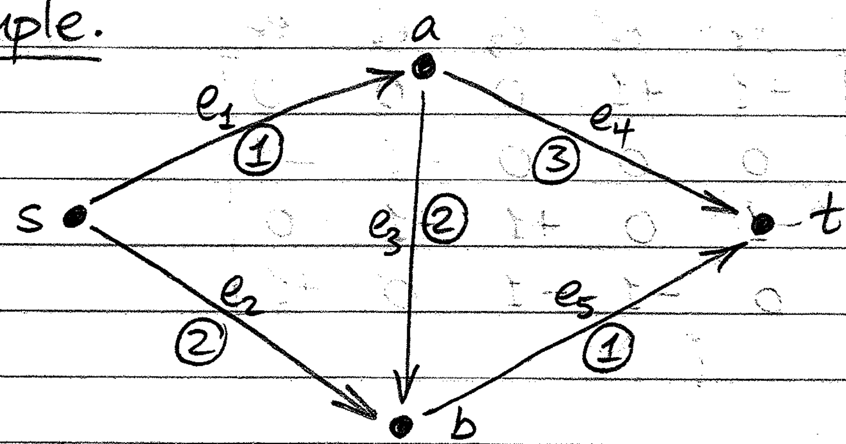
The Shortest-Path Problem and Its Dual [P&S §3.4]

Given: A directed graph (each edge has a direction) with a nonnegative weight on each edge, and a specified source node s and a specified terminal node t.

Objective: Find a (directed) path from s to t of minimum total weight.
- Interpreting the weights as distances, this is a shortest path from s to t.

Example.



Edge weights are circled.

Three paths from s to t in this graph:
1. s—a—t: total weight $1+3=4$
2. s—a—b—t: total weight $1+2+1=4$
3. s—b—t: total weight $2+1=3$

So the shortest path from s to t is s—b—t.

Node-arc incidence matrix.

For a directed graph with n nodes (vertices) and m arcs (directed edges), the node-arc incidence matrix is an $n \times m$ matrix A in which the rows correspond to nodes and the columns correspond to edges. The entry a_{ij} is

$$a_{ij} = \begin{cases} +1, & \text{if arc } e_j \text{ leaves node } i; \\ -1, & \text{if arc } e_j \text{ enters node } i; \\ 0, & \text{otherwise.} \end{cases}$$

For the preceding example:

$$A = \begin{array}{c} \\ s \\ t \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} +1 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & +1 & +1 & 0 \\ 0 & -1 & -1 & 0 & +1 \end{bmatrix} \end{array}$$

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LP formulation of shortest path

Idea: Send out a "flow" of 1 unit (of water, say) from s. The arc weights represent the per-unit cost of sending part (or all) of the flow through that arc. Node t will consume the flow. Every node other than s and t will conserve flow: the total flow going into the node must equal the total flow coming out. The minimum-cost way to do this will give us the shortest (i.e., least expensive) path from s to t.

Variables: One variable f_j for each arc e_j, indicating the flow to be sent throug that arc.

— Domain: $f_j \geq 0$ for all j.

Constraints:

- Node s must send out a flow of 1 unit:

$$\sum_{\substack{j \text{ such that} \\ e_j \text{ leaves node } s}} f_j - \underbrace{\sum_{\substack{j \text{ such that} \\ e_j \text{ enters node } s}} f_j}_{\text{incoming flow (if any)}} = 1$$

- Node t must consume a flow of 1 unit:

$$\sum_{\substack{j \text{ such that} \\ e_j \text{ enters node } t}} f_j - \sum_{\substack{j \text{ such that} \\ e_j \text{ leaves node } t}} f_j = 1$$

- All other nodes must conserve flow, i.e., total outgoing flow must equal total incoming flow:

$$\underbrace{\sum_{\substack{j \text{ such that} \\ e_j \text{ leaves node } i}} f_j}_{\text{total outgoing flow}} - \underbrace{\sum_{\substack{j \text{ such that} \\ e_j \text{ enters node } i}} f_j}_{\text{total incoming flow}} = 0 \quad \text{for all } i \text{ except } s, t$$

<u>Objective:</u> Minimize cost:

$$\min \sum_{j=1}^{m} c_j f_j$$

cost (weight) of arc e_j

For the example:

$$\begin{array}{llllllll}
\min & f_1 & + 2f_2 & + 2f_3 & + 3f_4 & + f_5 & & \\
\text{s.t.} & f_1 & + f_2 & & & & = 1 & [\text{node } s] \\
 & & & & f_4 & + f_5 & = 1 & [\text{node } t] \\
 & -f_1 & & + f_3 & + f_4 & & = 0 & [\text{node } a] \\
 & & - f_2 & - f_3 & & + f_5 & = 0 & [\text{node } b]
\end{array}$$

All variables nonnegative.

Shortest path LP formulation — 2

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Note that if we negate the constraint for node t, then the left-hand side of every constraint represents the <u>net outflow</u> for a node:

$$
\begin{array}{lrcrcrcrcrcrl}
\min & f_1 & + & 2f_2 & + & 2f_3 & + & 3f_4 & + & f_5 & & & \\
\text{s.t.} & f_1 & + & f_2 & & & & & & & = & 1 & [\text{node } s] \\
 & & & & & & & -f_4 & - & f_5 & = & -1 & [\text{node } t] \\
 & -f_1 & & & + & f_3 & + & f_4 & & & = & 0 & [\text{node } a] \\
 & & & -f_2 & - & f_3 & & & + & f_5 & = & 0 & [\text{node } b]
\end{array}
$$

All variables nonnegative.

And the coefficient matrix for this LP is the node-arc incidence matrix for the directed graph.

So, in general, we have

$$
\begin{array}{ll}
\min & \overset{\text{vector of arc weights}}{c^T} f \\
\text{s.t.} & \underset{\text{node-arc incidence matrix}}{A} f = \left[\begin{array}{c} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{array}\right] \begin{array}{l} \leftarrow \text{Row } s \\ \leftarrow \text{Row } t \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{All other rows} \end{array} \\
 & f \geq 0.
\end{array}
$$

Next observe that the constraints are redundant — any $n-1$ of the constraints imply the remaining one. So we can remove any single constraint without changing the feasible region. Let's remove the constraint for node t so that the RHS of every constraint is nonnegative:

$$
\begin{array}{lrcrcrcrcrcll}
\min & f_1 & + & 2f_2 & + & 2f_3 & + & 3f_4 & + & f_5 & & & \\
\text{s.t.} & f_1 & + & f_2 & & & & & & & = & 1 & \text{[node s]} \\
 & -f_1 & & & + & f_3 & + & f_4 & & & = & 0 & \text{[node a]} \\
 & & - & f_2 & - & f_3 & & & + & f_5 & = & 0 & \text{[node b]}
\end{array}
$$

All variables nonnegative.

The dual of this LP is

$$
\begin{array}{lrcrcrcl}
\max & \pi_s & & & & & & \\
\text{s.t.} & \pi_s & - & \pi_a & & & \le & 1 \\
 & \pi_s & & & - & \pi_b & \le & 2 \\
 & & & \pi_a & - & \pi_b & \le & 2 \\
 & & & \pi_a & & & \le & 3 \\
 & & & & & \pi_b & \le & 1
\end{array}
$$

All variables unrestricted.

Shortest path LP formulation — 3

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Initial simplex tableau:

π_s^+	π_s^-	π_a^+	π_a^-	π_b^+	π_b^-	s_1	s_2	s_3	s_4	s_5	z	RHS
-1	1	0	0	0	0	0	0	0	0	0	1	0
1	-1	-1	1	0	0	1	0	0	0	0	0	1
1	-1	0	0	-1	1	0	1	0	0	0	0	2
0	0	1	-1	-1	1	0	0	1	0	0	0	2
0	0	1	-1	0	0	0	0	0	1	0	0	3
0	0	0	0	1	-1	0	0	0	0	1	0	1

Optimal simplex tableau:

optimal values of primal vars

π_s^+	π_s^-	π_a^+	π_a^-	π_b^+	π_b^-	s_1	s_2	s_3	s_4	s_5	z	RHS
0	0	0	0	0	0	0	1	0	0	1	1	3
1	-1	0	0	0	0	0	1	0	0	1	0	3
0	0	1	-1	0	0	-1	1	0	0	1	0	2
0	0	0	0	0	0	1	-1	1	0	0	0	1
0	0	0	0	0	0	1	-1	0	1	-1	0	1
0	0	0	0	1	-1	0	0	0	0	1	0	1

Optimal dual solution:

$$\pi_s = 3, \quad \pi_a = 2, \quad \pi_b = 1, \quad [\pi_t = 0].$$

(We deleted the node t constraint, which is equivalent to deleting the π_t column in the dual. So π_t does not have a chance to become basic, so its value remains at 0.)

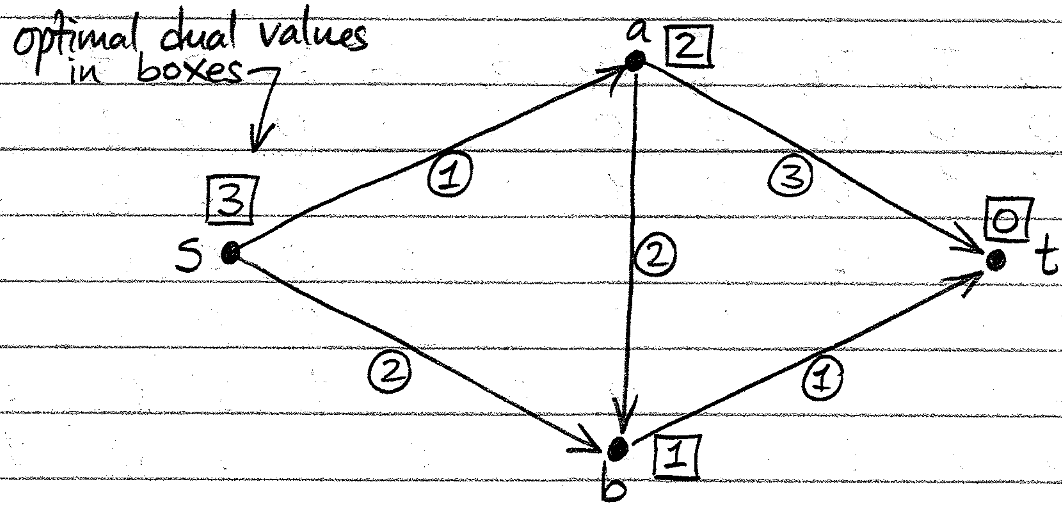
Optimal primal solution: (from obj row of optimal tableau)

$$f_1 = 0, \ f_2 = 1, \ f_3 = 0, \ f_4 = 0, \ f_5 = 1.$$ ← indicate arcs in shortest path

Optimal objective value: 3. ← length of shortest path

What do the dual variables mean?

Dual variables are associated with nodes:



Dual LP [with π_t column included this time]:

$$
\begin{array}{lllllll}
\max & \pi_s - \pi_t & & & & & \\
\text{s.t.} & \pi_s & & -\pi_a & & \leq 1 \\
 & \pi_s & & & -\pi_b & \leq 2 \\
 & & & \pi_a & -\pi_b & \leq 2 \\
 & & -\pi_t & +\pi_a & & \leq 3 \\
 & & -\pi_t & & +\pi_b & \leq 1
\end{array}
$$

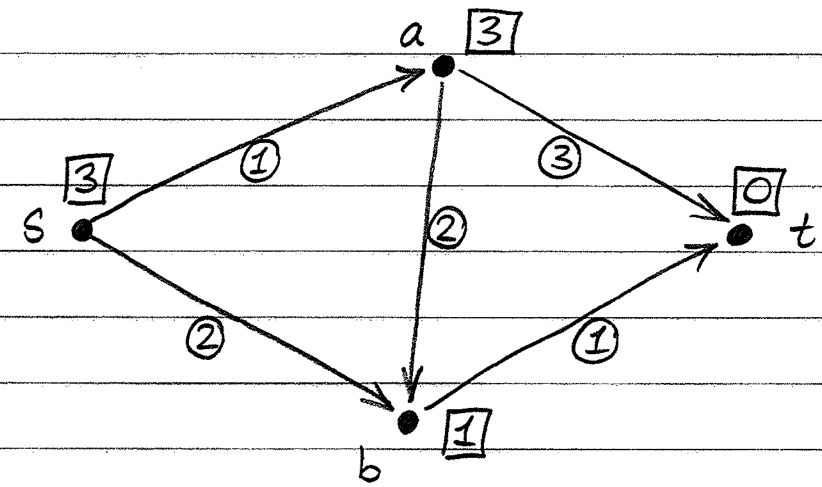
All variables unrestricted.

One interpretation: Dual values are "heights" of nodes. The flow of water through an arc equals the difference in heights of its endpoints. Flows cannot exceed capacities. Maximize drop in height from s to t.

— Alternatively: Dual values are "temperatures," arc capacities are maximum allowable temperature differences, maximize temperature difference between s and t.

Shortest path LP formulation — 4

But this optimal dual solution is not unique. Here is another optimal dual solution:



$\pi_s = 3$
$\pi_a = 3$
$\pi_b = 1$
$\pi_t = 0$

These numbers have a more interesting interpretation:

The value π_i is the shortest distance from i to t.

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Primal–dual algorithm [P&S Chap. 5]

Goal: Solve a min LP in standard form:

$$\begin{array}{ll} \min & z = c^T x \\ \text{s.t.} & Ax = b \geq 0 \\ & x \geq 0 \end{array} \qquad [\text{Primal LP}]$$

Note that we are assuming $b \geq 0$ here. If necessary, multiply some of the equations by -1 in order to make this true.

Dual LP:
$$\begin{array}{ll} \max & w = b^T \pi \\ \text{s.t.} & A^T \pi \leq c \\ & \pi \text{ unrestricted.} \end{array}$$

Recall complementary slackness: If x and π are feasible solutions to the primal and dual, respectively, then they are optimal if and only if

- for all i, either $\pi_i = 0$ or $\underbrace{a_i x = b_i}_{i\text{th primal constraint is tight}}$ (or both);

- for all j, either $x_j = 0$ or $\underbrace{(A_j)^T \pi = c_j}_{j\text{th dual constraint is tight.}}$ (or both).

Note that because the constraints in the primal are equalities, all primal constraints will always be tight for any feasible solution x, so the first complementary slackness condition will be satisfied automatically with no conditions on π.

Therefore, if we have a feasible solution π to the dual, and we can find a feasible solution x to the primal such that

$$x_j = 0 \quad \text{whenever} \quad (A_j)^T \pi < c_j,$$

then all the complementary slackness conditions will be satisfied, and we can conclude that both π and x are optimal.

This is the idea of the primal-dual algorithm:

1. Start with a feasible dual solution π.

2. Search for a feasible primal solution x such that $x_j = 0$ whenever $(A_j)^T \pi < c_j$.

3. If we succeed in step 2, we're done.

4. Otherwise, use the "best" solution found in step 2 (i.e., "closest to being feasible") to adjust the dual solution, and repeat.
   (improve)