Overview of the transportation algorithm

1. Form an initial basic feasible solution \( x_{ij} \) to the primal LP.

2. Use complementary slackness to find a corresponding solution \( v_i, w_j \) to the dual LP.

   (By complementary slackness, when \( x_{ij} \) is basic, we should have \( v_i + w_j = c_{ij} \).)

3. Compute test values \( c_{ij} - v_i - w_j \).

   (Test values for \( i, j \) corresponding to basic \( x_{ij} \) will always be zero, because of step 2. So we just need test values for nonbasic \( x_{ij} \).)

4. If all test values are nonnegative, STOP: current solution \( x_{ij} \) is optimal.

   (By second theorem.)

5. Otherwise, choose a negative test value (as a rule of thumb, the most negative one) and increase the corresponding \( x_{ij} \) as much as possible to decrease the cost. This is called a pivot. (Example shortly.) Then go back to step 2.
The transportation tableau

Rows correspond to origins.
Columns correspond to destinations.

Example (same as previously):

<table>
<thead>
<tr>
<th>Supply</th>
<th>Demand</th>
<th>Per-unit costs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>W  X  Y  Z</td>
</tr>
<tr>
<td>A</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>Y  7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Z  8</td>
</tr>
</tbody>
</table>

Blank transportation tableau with this information:

<table>
<thead>
<tr>
<th>W:8</th>
<th>X:12</th>
<th>Y:7</th>
<th>Z:8</th>
</tr>
</thead>
<tbody>
<tr>
<td>A:10</td>
<td>12</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>B:15</td>
<td>15</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>C:10</td>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Squares in this grid correspond to variables in the LP, so we will have basic and nonbasic squares corresponding to basic and nonbasic variables.
First step: Get an initial basic feasible solution.

Method:
- Satisfy each demand one by one, from left to right.
- For each demand, take units from each supply, top to bottom, until the demand is met; but once a supply has been exhausted, it cannot be used to satisfy later demands.
- If necessary, add one or more basic squares with a value of zero. (More about this later.)

For this example, our initial basic feasible solution is:

<table>
<thead>
<tr>
<th></th>
<th>W: 8</th>
<th>X: 12</th>
<th>Y: 7</th>
<th>Z: 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: 10</td>
<td>8</td>
<td>12</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>B: 15</td>
<td>15</td>
<td>2</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>C: 10</td>
<td>7</td>
<td>4</td>
<td>6</td>
<td>11</td>
</tr>
</tbody>
</table>

Basic squares are shaded. Number in lower left corner is value of corresponding basic variable $x_{ij}$, i.e., number of units to be sent from origin $i$ to destination $j$.

Objective value (total cost):

$$8(12) + 2(7) + 10(8) + 5(10) + 2(6) + 8(11) = \$344.$$
Transportation tableau — (2)

Note: Suppose we make a bipartite graph with vertices representing origins on the left and vertices representing destinations on the right, with an edge joining origin i to destination j if and only if the variable $x_{ij}$ is basic:

```
A --- W
   /   |
  X   Y
```

Observe that this graph is a tree: it is connected (there is a path between any two vertices) and it has no cycles. (This particular graph happens to be a path graph, but that will not be true in general.)

This is an important fact. Basic solutions for the transportation problem will always correspond to trees.

* If necessary, include one or more basic squares with value zero in order to get a tree.
Now that we have an initial basic feasible solution, we use complementary slackness to find a corresponding solution \( \{v_i, w_j\} \) to the dual LP. Complementary slackness implies that when \( x_{ij} \) is basic, we should have \( v_i + w_j = c_{ij} \).

From complementary slackness:

\[
\begin{align*}
(x_{AW}) & \quad v_1 + w_1 = 12 \\
(x_{AX}) & \quad v_1 + w_2 = 9 \\
(x_{BX}) & \quad v_2 + w_2 = 8 \\
(x_{BY}) & \quad v_2 + w_3 = 10 \\
(x_{CY}) & \quad v_3 + w_3 = 6 \\
(x_{CZ}) & \quad v_3 + w_4 = 11
\end{align*}
\]

We have six equations in seven unknowns, so we have one free variable—we can set any one of these variables to any value we like, and then solve for the remaining six.
For simplicity, let's set $v_1 = 0$. Solving for the remaining six dual variables, we get:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
<td>2</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>8</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>11</td>
</tr>
</tbody>
</table>

$w_1 = 12 \quad w_2 = 9 \quad w_3 = 11 \quad w_4 = 16$

Now, for the nonbasic squares, we calculate test values, $c_{ij} - v_i - w_j$, and write them in the lower right corner:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8</td>
<td>2</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>8</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>11</td>
</tr>
</tbody>
</table>

$w_1 = 12 \quad w_2 = 9 \quad w_3 = 11 \quad w_4 = 16$

We have negative test values, so this solution is not optimal. Choose the most negative ($-9$ in AZ) and pivot there.
Pivoting. The test value $-9$ in the $A_2$ square indicates that if we make $X_{A_2}$ positive (by bringing it into the basis) then we will decrease our total cost. So we want to increase the value of $X_{A_2}$ from its current value of zero to some new value $t$.

But unless we adjust the values of the other $x_{ij}$'s, we will mess up our row and column sums. The total of the $x_{ij}$'s in the $Z$ column must be 8 (to satisfy $Z$'s demand), so if we increase $X_{A_2}$ to $t$ then we need to decrease $X_{C_2}$ to the value $8-t$.

Now that adjustment messed up the total for the $C$ row, so we need to increase the value of $X_{C_2}$ to $2+t$.

Following this chain of reasoning:

- (Y column) Decrease $x_{BY}$ to $5-t$.
- (B row) Increase $x_{BX}$ to $10+t$.
- (X column) Decrease $X_{AX}$ to $2-t$.

This last adjustment also fixed the $A$ row sum that we messed up by increasing $X_{A_2}$ at the beginning. So now we're all good.
Summary of these adjustments:

Note that these adjustments follow a circuit of "alternating-direction rook's moves" (alternating vertical and horizontal motions) that, except for the pivot square AZ, turns 90° only on basic squares. The adjustments alternate $+t$, $-t$, $+t$, $-t$, ... at the "corners" of this circuit, starting with $+t$ at the pivot square.

There will always be exactly one such circuit in the tableau for any given pivot square. This is because the basic squares correspond to the edges of a tree, and adding one edge (i.e., the pivot square) to a tree creates a unique cycle:
Next question: What should \( t \) be?

Every unit increase to the value of \( x_{42} \) will decrease our total cost by \$9\) (that's what the test value means), so we want to make \( t \) as large as possible.

But if \( t \) is too large, the values of \( x_{31} \), \( x_{51} \), and \( x_{72} \), which are \( 2-t \), \( 5-t \), and \( 8-t \), respectively, will become negative, which violates their domains. The first to become negative will be \( x_{31} = 2-t \), so the greatest we can make \( t \) is 2.

(Note that \( x_{31} = 10+t \) and \( x_{51} = 2+t \) do not place restrictions on \( t \), because they will not become negative.)
Transportation pivoting - (3)

So, taking $t = 2$, our new basic feasible solution is

$$AX: \text{fell out of basis (} x_{AX} \text{ became zero)}$$

$$AZ: \text{new basic square}$$

Objective value: $8(12) + 2(7) + 12(8) + 3(10) + 4(6) + 6(11) = \underline{\$326}$

(Improved)

$x_{AW}$ is still 8 because $AW$ was not part of the pivot circuit.

Compute dual variables and test values:

$$W_1 = 12 \quad W_2 = 0 \quad W_3 = 2 \quad W_4 = 7$$
Still have negative test values, so still not optimal. Pivot on CW. Pivot circuit goes CW−AW−AZ−CZ−CW:

```
<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>8-t</td>
<td></td>
<td>2+t</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>t</td>
<td></td>
<td></td>
<td>6-t</td>
</tr>
</tbody>
</table>
```

So largest possible value of t is 6.
Next tableau:

```
<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
<td>9</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>8</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>0</td>
<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>
```

\[ V_1 = 0 \]
\[ V_2 = -1 \]
\[ V_3 = -5 \]

\[ W_1 = 12 \]
\[ W_2 = 9 \]
\[ W_3 = 11 \]
\[ W_4 = 7 \]

All test values are nonnegative, so this solution is optimal:

2 units A→W, 8 units A→Z, 12 units B→X,
3 units B→Y, 6 units C→W, 4 units C→Y.

Optimal objective value (cost): \[2(12) + 8(7) + 12(8) + 3(10) + 6(7) + 4(6) = \$272.\]