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Example. The dual of the dual is the primal, so if we want to find <u>lower bounds</u> for a (dual) min LP, using linear combinations of constraints, the best multipliers to use are the optimal values of the primal max LP. Example from earlier:

PRIMAL

$$\begin{aligned} \max\ & 8x_1+5x_2+6x_3 \\ \text{s.t.}\ & x_1+x_2+x_3 \le 8 \\ & 4x_1+2x_2-x_3 \le 7 \\ & x_1\ge 0,\ x_2\ge 0,\ x_3\ge 0 \end{aligned}$$

Optimal primal solution:
$x_1=3,\ x_2=0,\ x_3=5$

Optimal objective value:
$8(3)+5(0)+6(5)=54$

DUAL

$$\begin{aligned} \min\ & 8y_1+7y_2 \\ \text{s.t.}\ & y_1+4y_2 \ge 8 \\ & y_1+2y_2 \ge 5 \\ & y_1-y_2 \ge 6 \\ & y_1\ge 0,\ y_2\ge 0 \end{aligned}$$

Optimal dual solution:
$y_1=6.4,\ y_2=0.4$

Optimal objective value:
$8(6.4)+7(0.4)=54$

Upper bound using optimal dual solution as multipliers:

$$\begin{array}{rl} 6.4(x_1+x_2+x_3) & \le 6.4(8) \\ +\,0.4(4x_1+2x_2-x_3) & \le 0.4(7) \\ \hline 8x_1+7.2x_2+6x_3 & \le 54 \end{array}$$

so $z^* \le 54$.

Lower bound using optimal primal solution as multipliers:

$$\begin{array}{rl} 3(y_1+4y_2) & \ge 3(8) \\ 0(y_1+2y_2) & \ge 0(5) \\ +\,5(y_1-y_2) & \ge 5(6) \\ \hline 8y_1+7y_2 & \ge 54 \end{array}$$

so $w^* \ge 54$
(where w^* is the optimal objective value to the dual).

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<u>Weak duality theorem.</u>
If x and y are feasible solutions to a max LP and its dual min LP, respectively, then the objective value of x is less than or equal to the objective value of y.

<u>Proof.</u> By construction of the dual. ☐

<u>Corollary.</u> If one of the LPs in a primal-dual pair is unbounded, then the other must be infeasible.

<u>Recall:</u> There are three possibilities for an LP:

1. It has a (finite) optimal feasible solution;
2. It is unbounded;
3. It is infeasible.

For a primal-dual pair: [see P&S pp. 70–71]

PRIMAL \ DUAL	Finite optimum	Unbounded	Infeasible
Finite optimum	POSSIBLE	IMPOSSIBLE (above Corollary)	IMPOSSIBLE (strong duality) next
Unbounded	IMPOSSIBLE (above Corollary)	IMPOSSIBLE (above Corollary)	POSSIBLE
Infeasible	IMPOSSIBLE (strong duality)	POSSIBLE	POSSIBLE

Strong duality theorem. [P&S Thm 3.1]
If an LP has an optimal solution, then so does its dual, and the optimal objective values are equal.

Proof: See P&S.

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Theorem [Complementary slackness — P&S § 3.2]
Consider a primal–dual pair:

PRIMAL	DUAL
$\max\ c^T x$	$\min\ b^T y$
s.t. $Ax \lesseqgtr b$	s.t. $A^T y \lesseqgtr c$
$x \lesseqgtr 0$	$y \lesseqgtr 0$

[The $\lesseqgtr$ sign in the constraints represents the sequence of $\leq$, $\geq$, or $=$ signs between the left-hand sides and the right-hand sides. The domains $x \lesseqgtr 0$ and $y \lesseqgtr 0$ mean some set of domains of the form $x_i \geq 0$, $x_i \leq 0$, or x_i unrestricted.]

If x is a feasible solution to the primal and y is a feasible solution to the dual, then they are <u>optimal</u> if and only if

$$\underbrace{(b_i - a_i x)}_{\substack{\text{difference between} \\ \text{RHS and LHS of} \\ \text{ith constraint in} \\ \text{primal.}}} y_i = 0 \quad \text{for all } i$$

and

$$\underbrace{(c_j - A_j^T y)}_{\substack{\text{difference between} \\ \text{RHS and LHS of} \\ \text{jth constraint in} \\ \text{dual.}}} x_j = 0 \quad \text{for all } j.$$

In other words, complementary slackness says that for optimal solutions to a primal — dual pair:

- For each <u>constraint</u> in one problem and the corresponding <u>variable</u> in the other problem, either

  — the constraint is <u>tight</u> (a.k.a. <u>binding</u>) (two sides of constraint are equal)

  or

  — the variable is <u>zero</u>

  (or <u>both</u>).

<u>Proof</u>: See P&S.

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Using complementary slackness to find an optimal dual solution.

Example.
$$\begin{aligned} \max\ & 3x_1 + 5x_2 + 4x_3 \\ \text{s.t.}\ & x_1 + x_2 + x_3 \le 10 \\ & 2x_1 + 3x_2 \ge 6 \\ & x_1 + x_2 - x_3 = 4 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{aligned}$$

Optimal solution has $x_1 = 0$, $x_2 = 7$, $x_3 = 3$. Find the optimal solution to the dual min LP.

First, write the dual:

$$\begin{aligned} \min\ & 10y_1 + 6y_2 + 4y_3 \\ \text{s.t.}\ & y_1 + 2y_2 + y_3 \ge 3 \\ & y_1 + 3y_2 + y_3 \ge 5 \\ & y_1 - y_3 \ge 4 \\ & y_1 \ge 0,\ y_2 \le 0,\ y_3 \text{ unrestricted.} \end{aligned}$$

Now apply complementary slackness.

With primal constraints and dual variables:

1. Either $\underbrace{x_1 + x_2 + x_3 = 10}_{\text{TRUE: } 0+7+3=10}$ or $y_1 = 0$ (or both).

   – So no conclusion about y_1 here.

2. Either $\underbrace{2x_1 + 3x_2 = 6}_{\text{FALSE: } 2(0)+3(7)=21.}$ or $y_2 = 0$ (or both).

   — Therefore we must have $\boxed{y_2 = 0.}$

3. Either $\underbrace{x_1 + x_2 - x_3 = 4}_{\text{TRUE: } 0+7-3=4}$ or $y_3 = 0$ (or both).

   — Note that equality constraints will never yield useful information via complementary slackness.

With dual constraints and primal variables:

1. Either $y_1 + 2y_2 + y_3 = 3$ or $\underbrace{x_1 = 0}_{\text{TRUE}}$ (or both).

2. Either $y_1 + 3y_2 + y_3 = 5$ or $\underbrace{x_2 = 0}_{\text{FALSE: } x_2=7}$ (or both).

   — So we know $\boxed{y_1 + 3y_2 + y_3 = 5.}$

3. Either $y_1 - y_3 = 4$ or $\underbrace{x_3 = 0}_{\text{FALSE: } x_3=3.}$ (or both).

   — So we know $\boxed{y_1 - y_3 = 4.}$

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Using complementary slackness — 2

Therefore we know $y_2 = 0$ and we have the system

$$\begin{aligned} y_1 + 3y_2 + y_3 &= 5 \\ y_1 \qquad\quad - y_3 &= 4 \end{aligned}$$

which becomes (since $y_2 = 0$)

$$\begin{aligned} y_1 + y_3 &= 5 \\ y_1 - y_3 &= 4. \end{aligned}$$

Solution to this system: $y_1 = \frac{9}{2}$, $y_3 = \frac{1}{2}$.

Therefore the optimal dual solution is

$$\boxed{y_1 = \frac{9}{2}, \quad y_2 = 0, \quad y_3 = \frac{1}{2}.}$$

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Dual solution in the optimal simplex tableau. [P&S §3.5]

Thm. (If the objective row of the simplex tableau has the coefficient 1 for z, then) the optimal values for the variables in the dual appear in the objective row of the optimal simplex tableau in the columns for the slack and artificial variables for the corresponding primal constraints.

— The signs in the tableaux in P&S are flipped from the convention in lecture (their objective rows have coefficient -1 for z), so for their tableaux you need to flip the sign.

— This provides a use for artificial variable columns beyond Phase I. So keep artificial columns through Phase II, carrying them through pivots. Just don't pivot in the artificial columns themselves, because the whole point of Phase I was to kick the artificial variables out of the basis!

Proof of Thm. See P&S.

Example.

PRIMAL

$$\begin{aligned} \max\ & 3x_1 + 5x_2 + 4x_3 \\ \text{s.t.}\ & x_1 + x_2 + x_3 \le 10 \\ & 2x_1 + 3x_2 \ge 6 \\ & x_1 + x_2 - x_3 = 4 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{aligned}$$

DUAL

$$\begin{aligned} \min\ & 10y_1 + 6y_2 + 4y_3 \\ \text{s.t.}\ & y_1 + 2y_2 + y_3 \ge 3 \\ & y_1 + 3y_2 + y_3 \ge 5 \\ & y_1 - y_3 \ge 4 \\ & y_1 \ge 0,\ y_2 \le 0,\ y_3 \text{ unrestricted} \end{aligned}$$

Optimal simplex tableau for primal max LP:

x_1	x_2	x_3	$\overset{y_1}{\downarrow}$ s_1	p_2	$\overset{y_2}{\downarrow}$ a_2	$\overset{y_3}{\downarrow}$ a_3	z	RHS
2	0	0	9/2	0	0	1/2	1	47
0	0	1	1/2	0	0	−1/2	0	3
1	1	0	1/2	0	0	1/2	0	7
1	0	0	3/2	1	−1	3/2	0	15

slack column (s_1) — artificial columns (a_2, a_3)

So the optimal dual solution is $y_1 = \frac{9}{2}$, $y_2 = 0$, $y_3 = \frac{1}{2}$.

— This matches what we found earlier via complementary slackness.