1 June.

**Defn.** If a basic solution is feasible, it is a <u>basic feasible solution</u> (bfs).

[P&S §1.5, 2.3]

## Convex combinations and extreme points.

**Defn.** Let $x, y \in \mathbb{R}^n$. A <u>convex combination</u> of $x$ and $y$ is a point (i.e., vector) of the form

(Weighted average.)

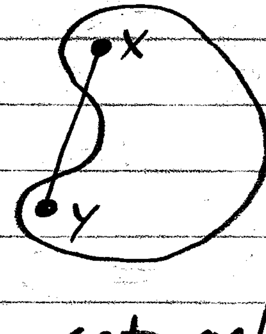
$$z = \lambda x + (1-\lambda) y,$$

where $\lambda$ is a scalar in the interval $[0,1]$. If $0 \lneq \lambda \lneq 1$, $z$ is a <u>strict</u> convex combination.

- Note that the set of all convex combinations of $x$ and $y$ forms the line segment joining $x$ and $y$.

**Defn.** A set $S \subseteq \mathbb{R}^n$ is <u>convex</u> if for all $x, y \in S$, all convex combinations of $x$ and $y$ are also in $S$.

<u>Example:</u> Not convex $\rightarrow$




- Problem 4 on first problem set asks you to prove that the feasible region of an LP is convex.

Defn. An *extreme point* of a convex set $S$ is a point $x \in S$ such that if $x$ is expressed as a strict convex combination of two points $y$ and $z$ in $S$, i.e., if

$$x = \lambda y + (1-\lambda) z \quad \text{for some } 0 < \lambda < 1,$$

then $y = z$.

- Intuitively: An extreme point does not lie between two other points in the set.

- An extreme point of a polytope is a *corner*.

[P&S Thm 2.4]

Theorem. A solution $x \in \mathbb{R}^n$ is an extreme point of the feasible region of an LP (in std form) if and only if it is a basic feasible solution.

Proof. By reordering the components of $x$ (and the columns of the coefficient matrix $A$), we may assume WLOG that the first $r$ components of $x$ are nonzero and the remaining $n-r$ components are zero:

$$x_i \neq 0 \quad \text{for } 1 \leq i \leq r,$$

$$x_i = 0 \quad \text{for } r+1 \leq i \leq n.$$

Then $x$ is a bfs if and only if the first $r$ columns of $A$ are linearly independent.

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[bfs $\Leftrightarrow$ extreme point of feas. region]

(bfs $\Rightarrow$ extreme point.) Suppose $x$ is a bfs; then the first $r$ columns $\{A_1, \dots, A_r\}$ of $A$ are lin. independent. Let $x$ be written as a <u>strict</u> convex combination of two points $y$ and $z$ in the feasible region:

$$x = \lambda y + (1-\lambda) z$$

for some $\lambda \in (0,1)$. Since $x$, $y$, and $z$ are all feasible solutions, we know that

$$\begin{array}{ccc} Ax = b & Ay = b & Az = b \\ x \geq 0 & y \geq 0 & z \geq 0. \end{array}$$

For $r+1 \leq i \leq n$, we have

$$\underbrace{x_i}_{=0} = \underbrace{\lambda}_{>0} \underbrace{y_i}_{\geq 0} + \underbrace{(1-\lambda)}_{>0} \underbrace{z_i}_{\geq 0}$$

so $y_i = z_i = 0 = x_i$. Therefore,

$$\begin{aligned} A_1 x_1 + \cdots + A_r x_r &= Ax = b, \\ A_1 y_1 + \cdots + A_r y_r &= Ay = b, \\ A_1 z_1 + \cdots + A_r z_r &= Az = b. \end{aligned}$$

But by assumption $\{A_1, \dots, A_r\}$ are lin indep, so the representation of $b$ as a lin combo is unique, and hence $x = y = z$. So $x$ is an extreme point of the feasible region. ✓

(Not bfs $\Rightarrow$ not extreme point.) Suppose $x$ is not a bfs, so the first $r$ columns $\{A_1, \dots, A_r\}$ of $A$ are not linearly independent. Then there exist scalars $\alpha_1, \dots, \alpha_r$, not all zero, such that

$$A_1 \alpha_1 + \cdots + A_r \alpha_r = 0.$$

Let $\alpha$ be the vector $\alpha = [\alpha_1, \dots, \alpha_r, 0, \dots, 0]^T$, so that $A\alpha = 0$. Choose $\lambda > 0$ small enough that $\lambda |\alpha_i| \leq x_i$ for all $1 \leq i \leq r$.

[E.g., $\lambda = \min \left\{ \frac{x_i}{|\alpha_i|} : 1 \leq i \leq r, \ \alpha_i \neq 0 \right\}$.]

Let $w = x + \lambda\alpha$ and $\overline{w} = x - \lambda\alpha$. By our choice of $\lambda$, we have $w \geq 0$ and $\overline{w} \geq 0$. Also,

$$A w = A(x + \lambda\alpha) = Ax + \lambda A\alpha = b + 0 = b,$$
$$A\overline{w} = A(x - \lambda\alpha) = Ax - \lambda A\alpha = b - 0 = b.$$

So $w$ and $\overline{w}$ are feasible solutions, and $w \neq x$ and $\overline{w} \neq x$ because $\lambda > 0$ and $\alpha \neq 0$. But $x = \frac{1}{2} w + \frac{1}{2} \overline{w}$, so $x$ is not an extreme point of the feasible region. ✓

$\square$

1 June. [≈ P&S Thm 2.6]

**Theorem.** Let $x$ be a feasible solution to an LP. Then either there exists a bfs whose objective value is at least as good as that of $x$, or else the LP is unbounded.

**Sketch of proof via example.** (Farmer Brown)

$$\begin{array}{llcl}
\max & 40p + 120w & & \\
\text{s.t.} & p + w + s_1 & = & 100 \\
& p + 4w + s_2 & = & 160 \\
& 10p + 20w + s_3 & = & 1100 \\
& p \geq 0,\ w \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0. & &
\end{array}$$

A basic solution will have at most <u>three</u> nonzero components.

$x = [20, 35, 45, 0, 200]^T$ is feasible but not basic (too many nonzero components). We can decrease the number of nonzeroes as follows.

First, find a nonzero solution to the <u>homogeneous</u> system

$$\left.\begin{array}{lcl}
p + w + s_1 & = & 0 \\
p + 4w + s_2 & = & 0 \\
10p + 20w + s_3 & = & 0
\end{array}\right\} \text{RHS zero!}$$

having $s_2 = 0$ (to preserve the zero we already have).

<u>Idea</u>: Find a vector $\Delta x \neq 0$ to add to $x$ such that
- $A\Delta x = 0$, so $A(x+\Delta x) = b + 0 = b$;
- $c^T \Delta x \geq 0$, so obj. val. does not decrease;
- $x + \Delta x \geq 0$ and $x+\Delta x$ has one more zero than $x$.

Set $s_2 = 0$. Now we have three equations in four unknowns, so one free variable. Set $w = 1$, say, to guarantee a nonzero solution. So we have

$$\begin{aligned} p + 1 + s_1 \quad &= 0 \\ p + 4 \quad &= 0 \\ 10p + 20 \quad + s_3 &= 0 \end{aligned}$$

Solution is $p = -4$, $s_1 = 3$, $s_3 = 20$. So full solution to homogeneous system is $h = [-4, 1, 3, 0, 20]^T$.

By linearity, we can add any scalar multiple of $h$ to $x$, and the resulting vector $y = x + th$ (for $t \in \mathbb{R}$) will also satisfy the constraints $Ay = b$, because

$$Ay = A(x + th) = Ax + t(Ah) = b + 0 = b.$$

The resulting objective value will be

$$c^T y = c^T(x + th) = c^T x + t(c^T h).$$

Since we want this to be at least as good (i.e., large) as $c^T x$, we want $t(c^T h) \geq 0$. We have

$$c^T h = 40(-4) + 120(1) = -40,$$

so we will take $t < 0$. (Alternatively, use $-h$ instead of $h$ and take $t > 0$.)

Now, how large in magnitude can $t$ be?

$$y = x + th = [20 - 4t, 35 + t, 45 + 3t, 0, 200 + 20t]^T.$$

The components of $y$ cannot be negative, so the second, third, and fifth components impose the constraints

$$\left.\begin{aligned} 35 + t &\geq 0 \\ 45 + 3t &\geq 0 \\ 200 + 20t &\geq 0 \end{aligned}\right\} \implies \left\{\begin{aligned} t &\geq -35 \\ t &\geq -15 \\ t &\geq -10 \end{aligned}\right.$$

(Note that first component imposes no constraint if $t < 0$.)

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The strongest of these three constraints is $t \geq -10$.

Taking $t = -10$, then, we get

$$y = x + th = [20-4(-10), 35-10, 45+3(-10), 0, 200+20(-10)]^T$$

$$= [60, 25, 15, 0, 0]^T.$$

This solution $y$

- satisfies constraints, because we began with feasible solution $x$ and added a solution to the homogeneous system;
- has all nonnegative components;

} hence is feasible

- has objective value at least as large as that of $x$, because $t(c^T h) \geq 0$.

✓

<u>Note:</u> If we had no constraints on the magnitude of $t$, then we could make $t$ arbitrarily large in magnitude, thereby increasing the objective value indefinitely; so the LP would be unbounded (and the ray $x + th$ would be a certificate of this fact).

- Small catch: What if $c^T h = 0$? Well, then we can use <u>either</u> $h$ or $-h$, and at least one of these will place constraints on the magnitude of $t$. →

[≈ P&S Thm 2.6]

★ **Corollary.** If an LP has an optimal feasible solution, then it has an optimal basic feasible solution.

— This justifies the geometric observation we made earlier, that we need only consider the corners of the feasible region when seeking an optimal solution (to a bounded LP).

1 June

# The simplex tableau and pivoting.

[P&S §2.4, 2.5, 2.9] **Example.** (Farmer Brown again)

$$\begin{aligned} \max\ & 40p + 120w \\ \text{s.t.}\ & p + w + s_1 = 100 \\ & p + 4w + s_2 = 160 \\ & 10p + 20w + s_3 = 1100 \\ & p \ge 0,\ w \ge 0,\ s_1 \ge 0,\ s_2 \ge 0,\ s_3 \ge 0. \end{aligned}$$

Ignoring the objective for now, we can express the constraints with the augmented matrix

(Column labels) →

| $p$ | $w$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 100 |
| 1 | 4 | 0 | 1 | 0 | 160 |
| 10 | 20 | 0 | 0 | 1 | 1100 |

This is called a *tableau* (except that it's missing a row for the objective — later).

— P&S write the RHS column on the left-hand side of the tableau, presumably because certain implementation details may become simpler if RHS is column 0.

The coefficient matrix here has an "obvious" basis, having basic variables $\{s_1, s_2, s_3\}$. Corresponding basic solution is $p = 0$, $w = 0$, $s_1 = 100$, $s_2 = 160$, $s_3 = 1100$.

Suppose we want to choose a different basis, e.g., $\{w, s_1, s_3\}$. What is the corresponding basic solution? We know we have $p=0$ and $s_2=0$. To find values of $w$, $s_1$, and $s_3$, we solve the resulting system, which we can do via Gauss–Jordan elimination. Observe that the $s_1$ and $s_3$ columns are already columns of the identity matrix, so if we just make the $w$ column be $[0,1,0]^T$ then we can read off the solution. We can do this with row operations:

| p | w | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 100 |
| 1 | 4 | 0 | 1 | 0 | 160 |
| 10 | 20 | 0 | 0 | 1 | 1100 |

"Pivot on" this entry

Multiply row 2 by 1/4

| p | w | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 100 |
| 1/4 | 1 | 0 | 1/4 | 0 | 40 |
| 10 | 20 | 0 | 0 | 1 | 1100 |

This operation is called a <u>pivot</u>.

Subtract row 2 from row 1
Subtract 20(row 2) from row 3

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# Simplex tableau and pivoting — 2

| $p$ | $w$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|
| 3/4 | 0 | 1 | −1/4 | 0 | 60 |
| 1/4 | 1 | 0 | 1/4 | 0 | 40 |
| 5 | 0 | 0 | −5 | 1 | 300 |

Now the "obvious" basis is $\{w, s_1, s_3\}$, and the corresponding basic solution is clearly $p=0$, $w=40$, $s_1=60$, $s_2=0$, $s_3=300$.

## Observe:

- This basic solution happens to be feasible: all variables have nonnegative values. (The solution still satisfies the system $Ax=b$ because row operations do not change the set of solutions to a system.)

- One pivot caused a single variable ($w$) to enter the basis, and another single variable ($s_2$) to leave the basis.

- This pivot caused us to move from one corner of the feasible region ($p=0$, $w=0$) to an adjacent corner ($p=0$, $w=40$).

- Pivoting on an entry means using row operations to make that entry 1 and all other entries in its column 0. //