"Anatomy" of an LP.

Objective $\rightarrow \max 40p + 120w$

Constraints $\rightarrow$

- $p + w \leq 100$
- $p + 4w \leq 160$
- $10p + 20w \leq 1100$

Variable domains $\rightarrow [p \geq 0, w \geq 0]$

Terminology

- **Solution**: An assignment of values to variables.
- **Feasible solution**: A solution that satisfies all constraints (and domains).
- **Feasible region** (a.k.a. feasible set): The set of all feasible solutions.
- **Objective value**: The value of the objective function corresponding to a given solution.
- **Optimal (feasible) solution**: A feasible solution whose objective value is at least as "good" as that of any other feasible solution.
- **Optimal objective value**: The objective value of an optimal feasible solution.
More terminology.

- **Feasible LP**: An LP with at least one feasible solution.
- **Infeasible LP**: One that is not feasible.
- **Unbounded LP**: A feasible LP with no optimal feasible solution.

Solving an LP means attempting to find an optimal feasible solution (not just the optimal objective value!).

Possible outcomes:

- LP is infeasible.
- LP is feasible.
- LP is unbounded.
- LP has an optimal feasible solution.
  - LP has a unique optimal feasible solution.
  - LP has a nonunique optimal feasible solution (i.e., at least two).

**Objective value**

| Objective value | 0 | 4000 | 4800 | 5400 | 4800 |

**Solution**

- (0, 0)
- (100, 0)
- (40, 25)
- (60, 0)

**Max** \(40p + 120w\)

**S.t.**

- \(p + w \leq 100\)
- \(10p + 20w \leq 1100\)
- \(p \geq 0\)
- \(w \geq 0\)

**Inequality**

- \(40p + 120w = 6000\)
- \(40p + 120w = 3600\)

**Optimal solution**

- \((60, 25)\)
Defn. A level curve of a function $f$ is the curve defined by $f = K$ for some constant $K$.

Graphical solution process (for an LP with two variables):

1. Draw constraints.

2. Determine feasible region, and find coordinates of corners.

3. Evaluate objective function at each corner.

4. Choose the best solution(s).

Warning: If the feasible region is unbounded, the LP may also be unbounded. Be sure to consider this possibility. Drawing a couple level curves may help.

Note that the feasible region of a two-variable LP is a polygon (if bounded). In higher dimensions (i.e., more variables), feasible region is a polytope.
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Matrix form of LP. [P&S §2.1]

Farmer Brown: \[ \text{max} \ 40p + 120w \]
\[
\begin{align*}
\text{s.t.} & \quad p + w \leq 100 \\
& \quad p + 4w \leq 160 \\
& \quad 10p + 20w \leq 1100 \\
& \quad p \geq 0, \ w \geq 0.
\end{align*}
\]

We can write this LP in terms of matrices as follows:

\[
\begin{align*}
\text{Max} \ & \ c^T x \\
\text{s.t.} \ & \ Ax \leq b \\
& \ x \geq 0
\end{align*}
\]

where

\[
c = \begin{bmatrix} 40 \\ 120 \end{bmatrix}, \ x = \begin{bmatrix} p \\ w \end{bmatrix}, \ A = \begin{bmatrix} 1 & 1 \\ 10 & 20 \end{bmatrix}, \ b = \begin{bmatrix} 100 \\ 160 \\ 1100 \end{bmatrix}.
\]

Note that \( c^T x = 40p + 120w, \)

\[
A x = \begin{bmatrix} p + w \\ p + 4w \\ 10p + 20w \end{bmatrix}.
\]

Vector inequality \( Ax \leq b \) (and \( x \geq 0 \)) is to be interpreted componentwise.
Slack variables and standard form.

We can turn inequalities into equalities by adding slack variables to fill up the gap between the two sides:

\[
\begin{align*}
\text{max} & \quad 40p + 120w \\
\text{s.t.} & \quad p + w + s_1 = 100 \\
& \quad p + 4w + s_2 = 160 \\
& \quad 10p + 20w + s_3 = 1100 \\
\end{align*}
\]

\[p \geq 0, \quad w \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \quad s_3 \geq 0\]

To ensure original inequalities are satisfied.

Therefore, we may convert any LP to standard form:

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

Note: — The \(x\) vector here includes any necessary slack variables.
— P&S prefers writing LPs as minimization problems. For certain reasons, I will write LPs as maximization problems instead. Conversion between the two is easy: negate objective function.
Basic feasible solutions. [P&S §2.2, 2.3]

Suppose the coefficient matrix $A$ of an LP in standard form is an $m \times n$ matrix (i.e., $m$ rows, $n$ columns).

Assumption: $A$ has $m$ linearly independent columns (i.e., $A$ has rank $m$).

- Intuitively, this is equivalent to saying the LP has no "redundant" constraints—no (LHS of) constraint is a linear combo of any others.

Defn. A basis of $A$ is a linearly independent collection $B$ of $m$ columns of $A$:

$$B = \{ A_j_1, A_j_2, \ldots, A_j_m \}.$$

Alternatively, we can think of $B$ as an $m \times m$ nonsingular matrix $B = [A_j_i]$ formed by choosing $m$ linearly independent columns of $A$.

The basic solution corresponding to $B$ is a vector $x \in \mathbb{R}^m$ such that

- nonbasic variables $\rightarrow x_j = 0$ for $A_j \notin B$;
- basic variables $\rightarrow x_{j_k} = k\text{th component of } B^{-1}b$, $k = 1, \ldots, m$. 

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So, to find a basic solution \( x \):

1. Choose a basis \( B \), a set of \( m \) linearly independent columns of \( A \).

2. Set all components of \( x \) corresponding to columns not in \( B \) equal to zero. (These are the nonbasic variables.)

3. Solve the \( m \) resulting equations to determine the remaining components of \( x \). (These are the basic variables.)

**Example.** \( \max 40p + 120w \)

\[
\begin{align*}
\text{s.t.} & \quad p + w + s_1 = 100 \\
& \quad p + 4w + s_2 = 160 \\
& \quad 10p + 20w + s_3 = 1100 \\
& \quad p \geq 0, \quad w \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \quad s_3 \geq 0
\end{align*}
\]

Obvious basis is \( B = \{ A_3, A_4, A_5 \} \), corresponding to basic variables \( \{ s_1, s_2, s_3 \} \), because these columns of \( A \) form the identity matrix.

This yields basic solution \( (p, w, s_1, s_2, s_3) = (0, 0, 100, 160, 1100) \).

- Note this is a corner of the feasible region.

Another basis is \( B = \{ A_3, A_1, A_4 \} \), corresponding to basic variables \( \{ p, s_1, s_2, s_3 \} \). This yields the basic solution \( (p, w, s_1, s_2, s_3) = (110, 0, -10, 50, 0) \).

- Note that this solution is not feasible because \( s_1 < 0 \).

**Defn.** If a basic solution is feasible, it is a basic feasible solution ( bfs).