28 May

Combinatorial optimization overview

- Combinatorial optimization is the study of problems that involve a search for the "best" option among a (usually finite) set of choices.

- Example: Factory placement. A company has ten possible sites at which it can build factories. A factory can produce 100 units/month of output. Each site has a different building cost:

<table>
<thead>
<tr>
<th>Site</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Building cost:</td>
<td>$1.2M</td>
<td>$1.8M</td>
<td>$0.8M</td>
<td>$2.1M</td>
<td>...</td>
</tr>
</tbody>
</table>

The company has a list of customers with monthly demands:

<table>
<thead>
<tr>
<th>Customer:</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly demand:</td>
<td>12</td>
<td>16</td>
<td>23</td>
<td>7</td>
<td>10</td>
<td>...</td>
</tr>
</tbody>
</table>

Shipping costs per unit:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>From:</td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$4</td>
<td>$6</td>
<td>$13</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To:</td>
<td></td>
<td>10</td>
<td>8</td>
<td>9</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>3</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Where should factories be built to satisfy customer demand over 5 years at min cost?
Example: Bin packing.

List of items of various sizes:

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>9</td>
<td>...</td>
</tr>
</tbody>
</table>

Supply of bins, each having the same capacity (say, capacity 50).

How to place all items into bins, using the minimum number of bins, without exceeding bin capacity?

"Steel mill slab problem": Each item has a color, and the number of different colors in each bin can be no more than 2.
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Combinatorial optimization overview

- Example: Swim meets. Two teams.

- Each team can enter at most three swimmers in any one individual (non-relay) event.
- Each team can provide at most three “entries” in each relay event, where an entry is a group of four swimmers.
- Each swimmer can enter at most four events.
- Each swimmer can enter at most two individual events.
- In a relay event, a team cannot be awarded points for more than two finishing places.
- Coach has information about event times of swimmers, and limited information about opposing team.
- How to assign swimmers to events to maximize probability of winning?

[Nowak, Epelman, Pollock]
More examples:
- Project scheduling (CPM)
- Minimum spanning tree
- Traveling salesman
Linear programming—intro example.

- Farmer Brown is planning to raise potatoes and wheat.
- 100 acres of land available.
- 160 days of labor available.
- One acre of wheat will require 4 days of labor.
- One acre of potatoes will require 1 day of labor.
- $1100 available for start-up costs of planting and cultivating.
- $10 per acre to plant/cultivate potatoes.
- $20 per acre to plant/cultivate wheat.
- Expected revenue:
  - $40/acre for potatoes,
  - $120/acre for wheat.

- How many acres of each crop should be planted to maximize total revenue?
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LP formulation.

1. Identify variables and their domains.
   - Variables usually represent decisions to be made—quantities under your control.

   - Imagine a form letter describing the best course of action to take, the best solution. Write a sentence that could be used to describe any solution, using blanks in place of numbers. Those blanks are the variables.

   - Sometimes auxiliary variables can be helpful—this usually becomes evident when trying to formulate constraints or objective function.

   - Each variable must have a corresponding domain. In an LP, there are three possible domains:

     \[ \begin{align*}
     & x \geq 0, \\
     & x \leq 0, \\
     & x \text{ unrestricted.}
     \end{align*} \]

     By far the most common.
LP formulation.

2. Identify the objective.
   - What quantity are you aiming to maximize or minimize?

3. Write the objective function in terms of the defined variables.
   - Remember that the objective function must be linear.

4. Identify the constraints in the problem and express each one in terms of the defined variables.
   - Remember that every constraint must be a linear inequality (or equality). Inequality cannot be strict.
   - Linear means:
     - No raising variables to powers,
     - No multiplication of variables together,
     - No dividing by variables,
     - No functions like \( \sin() \), \( \sqrt{} \), etc. applied to variables.
LP formulation for Farmer Brown example.

Variables:

\[ p: \text{ number of acres of potatoes} \]
\[ w: \text{ number of acres of wheat} \]

Domains: \[ p \geq 0, \quad w \geq 0 \]

Objective: Maximize revenue.

Objective function: \[ 40p + 120w \]

Constraints:

\[ \text{Land (resource):} \quad p + w \leq 100 \]
\[ \text{Labor (resource):} \quad p + 4w \leq 160 \]
\[ \text{Capital (resource):} \quad 10p + 20w \leq 1100 \]

Written: Maximize \[ 40p + 120w \]
Subject to \[ p + w \leq 100 \] [land]
\[ p + 4w \leq 160 \] [labor]
\[ 10p + 20w \leq 1100 \] [capital]

\[ p \geq 0, \quad w \geq 0. \]
"Anatomy" of an LP.

Objective: \[ \text{max} \ 40p + 120w \]

Constraints: \[
\begin{align*}
p + w & \leq 100 \\
p + 4w & \leq 160 \\
10p + 20w & \leq 1100
\end{align*}
\]

Variable domains: \[ p \geq 0, \ w \geq 0 \]

Terminology:
- Solution: An assignment of values to variables.
- Feasible solution: A solution that satisfies all constraints (and domains).
- Feasible region (a.k.a. feasible set): The set of all feasible solutions.
- Objective value: The value of the objective function corresponding to a given solution.
- Optimal (feasible) solution: A feasible solution whose objective value is at least as "good" as that of any other feasible solution.
- Optimal objective value: The objective value of an optimal feasible solution.
More terminology.

- **Feasible LP**: An LP with at least one feasible solution.
- **Infeasible LP**: One that is not feasible.
- **Unbounded LP**: A feasible LP with no optimal feasible solution.

Solving an LP means attempting to find an optimal feasible solution (not just the optimal objective value!).

Possible outcomes:

- LP is infeasible.
- LP is feasible.

- LP is unbounded.
- LP has an optimal feasible solution.
  - LP has a unique optimal feasible solution.
  - LP has a nonunique optimal feasible solution (i.e., at least two).
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Solving an LP with Maple.

Farmer Brown:

\[ \begin{align*}
\text{max} & \quad 40p + 120w \\
\text{s.t.} & \quad p + w \leq 100 \quad \text{[land]} \\
& \quad p + 4w \leq 160 \quad \text{[labor]} \\
& \quad 10p + 20w \leq 1100 \quad \text{[capital]} \\
\end{align*} \]

\[ p \geq 0, \quad w \geq 0. \]

In Maple:

restart;

with(Optimization);

\[ f := (p, w) \rightarrow 40* p + 120* w; \]

Constraints := [ p+w\leq 100, p+4*w\leq 160, 10*p+20*w\leq 1100 ];

LPSolve( f(p,w), constraints, \text{'maximize'},
assume = nonnegative );

Output: [5400., [p=60., w=25.]]