Sensitivity analysis

Big picture

- **The setting:** You have solved a linear program with the simplex method, and you have an optimal tableau.
- **The question:** If one of the numbers in the LP were changed slightly, how would that change affect the optimal solution?
- **The answer:** Sensitivity analysis provides two kinds of information.
  - A maximum increase and a maximum decrease that can be made to the number in the LP before the change becomes a “major” change. (A change within these limits is a “minor” change.)
  - If the change is a “minor” change, sensitivity analysis gives the optimal solution and the optimal objective value for the changed LP.

Correspondences

- Every constraint has a corresponding slack, surplus, and/or artificial variable:
  - A ≤ constraint has a corresponding slack variable.
  - An = constraint has a corresponding artificial variable.
  - A ≥ constraint has a corresponding surplus variable and a corresponding artificial variable.
- Every constraint also has a corresponding dual variable.
- Every variable has a corresponding column in the simplex tableau.
- Every basic variable has a corresponding row in the simplex tableau.
- Note: Rows in the simplex tableau do not correspond to constraints!

Meanings of the numbers in the objective row

The numbers in the objective row of a simplex tableau can be interpreted in two different ways:

- In any tableau, a number in the objective row in a nonbasic column indicates how much the value of the objective function will change if the value of the corresponding variable is increased by 1, but the sign is flipped: a negative entry indicates an increase in the objective value, and a positive entry indicates a decrease.
  - This is why, in the simplex algorithm, we pivot on columns with negative entries in the objective row. A negative number means that giving the corresponding variable a positive value (i.e., making that column basic) will increase the value of the objective function.
  - Note: This interpretation of the entries in the objective row gives information about the effects of a change to the basis of the current solution, without changing the LP itself; it gives information about what would happen if you move from the current corner of the feasible region to a neighboring corner of the feasible region by changing which variables are basic (and thereby changing the values of the variables).
- In an optimal tableau, the numbers in the objective row in columns corresponding to slack and artificial variables (not surplus variables) are the optimal values of the corresponding dual variables.
  - The optimal value of a dual variable is the amount that the optimal objective value would change if the right-hand side of the corresponding constraint were increased by 1. (There is no sign flip here: a positive dual variable indicates an increase in the objective value, and a negative dual variable indicates a decrease.)
  - Note: This interpretation gives information about the effects of a change to the LP itself.
Right-hand side sensitivity analysis

When the proposed change to the LP is to the right-hand side of a constraint.

- If the constraint is tight in the optimal solution:
  1. Look at the column for the corresponding slack or artificial variable (not the surplus variable). Call this column the “working column.”
  2. Form quotients by dividing the entries in the right-hand column of the optimal tableau by the entries in the working column (not including the entries in the objective row). [Mnemonic: Use the right-hand column for right-hand side sensitivity analysis.]
  3. To find the maximum increase: Consider the quotients from step 2 that use negative entries from the working column. Flip the signs of those quotients to make them positive, and then choose the smallest.
    - If the working column has no negative entries, then there is no maximum increase, i.e., there is no limit to the amount the right-hand side of the constraint can be increased.
  4. To find the maximum decrease: Consider the quotients from step 2 that use positive entries from the working column. Choose the smallest such quotient.
    - If the working column has no positive entries, then there is no maximum decrease, i.e., there is no limit to the amount the right-hand side of the constraint can be decreased.
  5. Within the limits identified in steps 3 and 4, every unit increase in the right-hand side of the constraint will change the right-hand column of the optimal tableau by the corresponding value in the working column. (This includes the value of the objective function, for which the per-unit change is given by the value of the optimal dual variable, found at the bottom of the working column.) Every unit decrease in the right-hand side of the constraint will have the opposite effect.
    - So to get the entries in the right-hand column of the new optimal tableau, multiply the change in the right-hand side of the constraint by each entry in the working column, and add those products to the corresponding entries in the right-hand side of the original optimal tableau. (Do this for the objective row too, to get the new optimal objective value.) Now you can easily read off the values of the basic variables in the optimal solution to the changed LP.

- If the constraint is not tight in the optimal solution:
  - For a \( \leq \) constraint, the corresponding slack variable will be positive. The value of this slack variable is the maximum decrease for the right-hand side of the constraint. There is no maximum increase (i.e., there is no limit to the amount the right-hand side of the constraint can be increased).
  - For a \( \geq \) constraint, the corresponding surplus variable will be positive. The value of this surplus variable is the maximum increase for the right-hand side of the constraint. There is no maximum decrease (i.e., there is no limit to the amount the right-hand side of the constraint can be decreased).
  - Within the limits identified above, any change to the right-hand side of the constraint will have no effect to the objective value or the optimal solution (except the value of the corresponding slack or surplus variable).

If the right-hand side of a constraint is changed by the maximum amount indicated by sensitivity analysis, the optimal solution for the changed LP will be degenerate. (One of the numbers in the right-hand column of the optimal tableau will be 0.)
Objective function coefficient sensitivity analysis

When the proposed change to the LP is to a coefficient of a variable in the objective function. (Suppose the variable is \( x_i \).)

- If the variable \( x_i \) is basic in the optimal solution:
  1. Look at the row corresponding to \( x_i \). Call this row the “working row.”
  2. Form quotients by dividing the entries in the objective row of the optimal tableau by the entries in the working row, but ignore the following columns:
     - the basic columns, including the column for \( x_i \);
     - any columns for artificial variables; and
     - the right-hand column.
  [Mnemonic: Use the objective row for objective function coefficient sensitivity analysis.]
  3. To find the maximum increase: Consider the quotients from step 2 that use negative entries from the working row. Flip the signs of those quotients to make them positive, and then choose the smallest.
     - If the working row has no negative entries, then there is no maximum increase, i.e., there is no limit to the amount the coefficient can be increased.
  4. To find the maximum decrease: Consider the quotients from step 2 that use positive entries from the working row. Choose the smallest such quotient.
     - If the working row has no positive entries, then there is no maximum decrease, i.e., there is no limit to the amount the coefficient can be decreased.
  5. Within the limits identified in steps 3 and 4, every unit increase in the objective function coefficient of \( x_i \) will increase the value of the objective function by the optimal value of \( x_i \). Every unit decrease in the coefficient will have the opposite effect. The optimal solution will not change.
     - For example, if the optimal value of \( x_3 \) is 57 and its objective function coefficient is increased by 1, then the optimal objective value will increase by 57 (because every unit of \( x_3 \) is now making $1 more profit). If its objective function coefficient is increased by 2, then the optimal objective value will increase by \( 2 \times 57 = 114 \).

- If the variable \( x_i \) is nonbasic in the optimal solution:
  - The variable \( x_i \) does not have a positive value because it is not profitable enough. The entry in the objective row in the \( x_i \) column, which must be nonnegative (because the tableau is optimal), indicates the amount that the objective value would decrease for each unit increase to the value of \( x_i \). So the per-unit profit for \( x_i \) (i.e., the objective function coefficient of \( x_i \)) would need to increase by this amount in order to make \( x_i \) profitable enough to produce. Until the per-unit profit increases this much, \( x_i \) will still not be profitable enough, so the optimal solution will not change.
  - Therefore, the maximum increase to the objective function coefficient of \( x_i \) (before the optimal solution changes) is given by the value in the objective row in the \( x_i \) column. There is no maximum decrease, i.e., there is no limit to the amount the coefficient can be decreased.
  - Any change to the objective function coefficient of \( x_i \) that does not exceed the maximum increase identified above will have no effect on the optimal objective value or the optimal solution.

If an objective function coefficient is changed by the maximum amount indicated by sensitivity analysis, the optimal solution for the changed LP will be non-unique. (One of the numbers in the objective row of the optimal tableau, in a nonbasic column, will be 0.)