Part A: sets [70 points]

Part A1: definitions [25 points]

Question A1.1 [5 points]
Give the definition of set equality.

Question A1.2 [5 points]
Let $X$ be a set. Define its power set.

Question A1.3 [5 points]
Give the definition of an indexed union of sets.

Question A1.4 [5 points]
Give the definition of a partition.

Question A1.5 [5 points]
Give the definition of the Cartesian product of two sets.

Part A2: problems [45 points]

Question A2.1 [15 points]

(i) Find non-empty sets $A, B, C$ such that $(A \setminus B) \setminus C = A \setminus (B \setminus C)$

(ii) Find non-empty sets $A, B, C$ such that $(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$

Question A2.2 [15 points]

Let $A$ and $B$ be sets. Determine whether the claims (i) and (ii) are true or false. If true, prove it. If false, provide a counter-example.

Note: this problem is very similar but is not the same as one of the homework problems.

(i) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

(ii) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$

Question A2.3 [15 points]

Consider the sets $A$ and $B$ defined as:

$$A = \left\{ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \frac{x}{y} + \frac{y}{x} \geq 2 \right\}$$

$$B = \{ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid (x > 0 \land y > 0) \lor (x < 0 \land y < 0) \}$$

where $\mathbb{R}_+ = \mathbb{R} \setminus \{0\}$

(i) Prove or disprove: $B \subseteq A$

(ii) Prove or disprove: $A \subseteq B$
Part B: logic [70 points]

Part B1: definitions [25 points]

Question B1.1 [5 points]
Give the definition of a mathematical statement.

Question B1.2 [5 points]
Let $P$ and $Q$ be mathematical statements. Give the definition of “$P \vee Q$”

Question B1.3 [5 points]
Let $P$ and $Q$ be mathematical statements. Give the definition of “$P \implies Q$”

Question B1.4 [5 points]
Give the definition of logical negation.

Question B1.5 [5 points]
Give the definition of logical equivalence.

Part B2: problems [45 points]

Question B2.1 [15 points]
Let $P, Q, R$ be mathematical statements. Provide the truth table for $(R \implies P) \land (\neg Q)$.

Question B2.2 [15 points]
For any $z \in \mathbb{Z}$, define $E(z)$ to be “$z$ is even”. Prove that

$$
\forall z \in \mathbb{Z}, E(z) \iff E(z^3)
$$

Question B2.3 [15 points]
Prove that

$$
\forall r \in \mathbb{R}, 1 \leq r < 2 \iff \left( \forall x > 0, 1 - \frac{1}{x} < r < 2 \right)
$$
Test 1 - Solutions

Part A1

Question A1.1 (see quiz 1.1)
Let A and B be sets. We say that A and B are equal if \( A = B \) and \( B \subseteq A \).

Question A1.2
The power set of \( X \) is the set whose elements are the subsets of \( X \).

Question A1.3
Let \( I \) be a set and let \( \{ A_i \}_{i \in I} \) be a collection of sets. Then \( \bigcup_{i \in I} A_i = \{ x \mid x \in A_i \text{ for some } i \in I \} \).

Question A1.4 (see quiz 1.2)
Let \( A \) be a set. A partition of \( A \) is a collection of pairwise disjoint subsets of \( A \) covering \( A \).

Question A1.5 (see quiz 2.1)
Let \( A \) and \( B \) be sets. The Cartesian product of \( A \) and \( B \) is the set of ordered pairs \((a, b)\) where \( a \in A \) and \( b \in B \).
Question A2.1 (see HW1.1)
(i) Pick any non-empty sets $A, B,$ and $C$ such that $A \cap C = \emptyset$
e.g. $A = \{1\}$ and $C = \{2\},$ such that $(A \cap B) \cup C = \emptyset = A - (B \cap C)$
(ii) Pick any non-empty sets $A, B,$ and $C$ such that $A \cap C \neq \emptyset$,
e.g. $A = B = C = \{1\},$ such that $(A \cap B) \cup C = \emptyset \neq \{1\} = A - (B \cap C)$

Question A2.2 (see HW2.1 for a similar problem)
(a) is true.
Proof:
Pick any $S \subseteq \mathcal{P}(A \cap B),$ i.e. $S \subseteq A \cap B$.
Recall that we have proven in class that $A \cap B \subseteq A$,
and observe that similarly, $A \cap B \subseteq B$.
Therefore, by transitivity of set inclusion, $S \subseteq A$ and $S \subseteq B$.
This means that $S \subseteq \mathcal{P}(A)$ and $S \subseteq \mathcal{P}(B),$ i.e. $S \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.
(b) is true.
Proof:
Pick any $S \subseteq \mathcal{P}(A \cap B),$ i.e. $S \subseteq A$ and $S \subseteq B$.
Claim: $S \subseteq A \cap B$
Proof of claim: Pick any $x \in S$.
Since $S \subseteq A,$ it follows that $x \in A$,
and since $S \subseteq B,$ it follows that $x \in B$.
Therefore $x \in A \cap B$.
Since $S \subseteq A \cap B,$ we thus have that $S \subseteq \mathcal{P}(A \cap B)$.

Question A2.3
(a) It is true that $B \subseteq A$.
Pick any $(x, y) \in B,$ i.e. $(x > 0 \land y > 0) \lor (x < 0 \lor y < 0)$.
Crucially, either way $x \cdot y > 0$.
Therefore: $\frac{x}{y} + \frac{y}{x} \geq 2 \iff x \cdot y \left(\frac{x}{y} + \frac{y}{x}\right) \geq 2 \cdot x \cdot y$
$\iff x^2 + y^2 \geq 2 \cdot x \cdot y$
$\iff x^2 - 2 \cdot x \cdot y + y^2 \geq 0$
$\iff (x - y)^2 \geq 0$
Since $(x - y)^2 \geq 0,$ it follows that $\frac{x}{y} + \frac{y}{x} \geq 2,$ i.e. $(x, y) \in A$. 
(ii) It is true that $A \subseteq B$.
Recall that, as we have seen in class, $A \subseteq B \implies B^c \subseteq A^c$
where of course we are taking complements with respect to the
universal set $\mathbb{R}_\times \times \mathbb{R}_\times$.
Let us therefore prove that $B^c \subseteq A^c$.
Let $(x,y) \in B^c$, i.e. $(x < 0 \land y > 0) \lor (x > 0 \land y < 0)$.
Crucially, either way, $xy < 0$.
Therefore: $\frac{x}{y} + \frac{y}{x} \geq 2 \implies xy \left( \frac{x}{y} + \frac{y}{x} \right) > 2xy$
$\implies x^2 + y^2 > 2xy$
$\implies x^2 - 2xy + y^2 > 0$
$\implies (x - y)^2 > 0$
\[ \text{i.e. } (x, y) \in A^c \implies (x - y)^2 > 0 \]
\[ \implies x - y \neq 0 \text{ since it always holds that } (x - y)^2 > 0 \]
\[ \implies x \neq y \]
So all we have left to do is show that $x \neq y$.
This holds since $(x < 0 \land y > 0) \lor (x > 0 \land y < 0)$,
such that either $x > y$
or $y > x$
i.e. either way $x \neq y$. 
Part B1

Question B1.1 (see quiz 2.1)
A mathematical statement is a grammatically correct sentence, composed of English words & mathematical symbols, that has exactly one truth value.

Question B1.2
"P \lor Q" is false when both P and Q are false, and true otherwise.

Question B1.3 (see quiz 2.2)
"P \rightarrow Q" is false when P is true and Q is false, and true otherwise.

Question B1.4 (see quiz 2.2)
Let P be a mathematical statement or proposition. The negation of P is defined to have truth values opposite to the truth values of P.

Question B1.5
Let P and Q be mathematical statements or propositions. We say that P and Q are equivalent when they have the same truth values.
Part B2 Question B2.1 (see HW 2.2 for a similar problem)

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>R \Rightarrow P</th>
<th>\neg Q</th>
<th>(R \Rightarrow P) \land (\neg Q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Question B2.2 (see HW 2.2 for \(\Rightarrow\))

First let us show that \(\forall z \in \mathbb{Z}, \ E(z) \Rightarrow E(z^3)\).

Let \(z \in \mathbb{Z}\) be even, i.e. \(a \in \mathbb{Z}\) such that \(z = 2a\).
Then \(z^3 = (2a)^3 = 8a^3 = 2(4a^3)\), i.e. \(z^3\) is even.

Now let us show that \(\forall z \in \mathbb{Z}, \ E(z^3) \Rightarrow E(z)\)

\(\neg E(z) \Rightarrow \neg E(z^3)\)
\(z\) is odd \(\Rightarrow\) \(z^3\) is odd

Let \(z \in \mathbb{Z}\) be odd, i.e. \(3a \in \mathbb{Z}\) s.t. \(z = 2a + 1\).
Then \(z^3 = (2a+1)^3 = 8a^3 + 4a^2 + 2a + 1\)
\(= 2(4a^3 + 2a^2 + a) + 1\)
i.e. \(z^3\) is odd.

Question B2.3

Observe that it is enough to show that \(\forall x \in \mathbb{R}, \ 1 \leq x \Rightarrow (Vx > 0, 1 - \frac{1}{x} < x)\)

We start by showing that \(\forall x \in \mathbb{R}, \ 1 \leq x \Rightarrow (Vx > 0, 1 - \frac{1}{x} < x)\).

Pick any \(x \in \mathbb{R}\) such that \(1 \leq x\).

Then, for any \(x > 0\), we have that \(\frac{1}{x} > 0\) and hence \(\frac{1}{x} < 0\).
Therefore \(1 - \frac{1}{x} < 1 \leq x\), i.e. indeed \(1 - \frac{1}{x} < x\).
Now let us show that \( \forall x \in \mathbb{R}, (\forall x > 0, 1 - \frac{1}{x} < \pi) \Rightarrow 1 < \pi. \)

Hence, \( 1 > \pi \Rightarrow (\exists x > 0 \text{ such that } 1 - \frac{1}{x} > \pi) \)

Choose any \( x \in \mathbb{R} \) such that \( 1 > \pi \), and take \( x = \frac{1}{1-\pi} \).

Then clearly \( x \in \mathbb{R} \) and \( 1 - \frac{1}{x} = 1 - (1 - \pi) \cdot x \), i.e., \( 1 - \frac{1}{x} > \pi \),
so we only have to show that \( x > 0 \).

Observe that:

\[
\begin{align*}
\text{(a)} & \quad x > 0 \\
\text{(b)} & \quad 1 - \frac{1}{x} > 0 \\
& \quad 1 - x > 0 \\
& \quad 1 > x
\end{align*}
\]

Since \( 1 > \pi \) indeed holds, we are done.
Part A: induction [70 points]

Part A1: definitions [10 points]

Question A1.1 [5 points]
State the principle of strong mathematical induction.

Question A1.2 [5 points]
State the well-ordering principle.

Part A2: problems [60 points]

Question A2.1 [15 points]
Show that for every nonzero \( n \in \mathbb{N} \), \( 4^{n+1} + 5^{2n-1} \) is a multiple of 21.

Question A2.2 [15 points]
Show that for every nonzero \( n \in \mathbb{N} \)
\[
\sum_{i=1}^{n} \frac{i - 1}{i!} = \frac{n! - 1}{n!}
\]
Recall that for every nonzero \( n \in \mathbb{N} \),
\[
n! = \prod_{i=1}^{n} i = n \cdot (n - 1) \cdot \ldots \cdot 2 \cdot 1.
\]

Question A2.3 [15 points]
Denoted \((F_n)_{n \in \mathbb{N}}\), the Fibonacci numbers are defined via
\[
\begin{cases}
F_0 = 0 \\
F_1 = 1 \\
F_n = F_{n-1} + F_{n-2} & \text{for all } n \geq 2
\end{cases}
\]
Show that for every \( n \in \mathbb{N} \)
\[
\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}
\]

Question A2.4 [15 points]
Given \( a_1, a_2 \in \mathbb{Z} \), define \( a_n = 2a_{n-1} + 3a_{n-2} \) for every natural number \( n \geq 3 \). Determine if the statement below is true or false and prove it.

If \( a_1 \) and \( a_2 \) are odd then \( a_n \) is odd for every \( n \in \mathbb{N} \setminus \{0\} \).

PLEASE TURN OVER
Part B: relations [70 points]

Part B1: definitions [25 points]

Question B1.1 [5 points]
Let \( x, y \in \mathbb{Z} \). Give the definition of ‘\( x \) divides \( y \)’.

Question B1.2 [5 points]
Give the definition of a relation.

Question B1.3 [5 points]
Give the definition of an equivalence class.

Question B1.4 [5 points]
Give the definition of a partial order.

Question B1.5 [5 points]
Give the definition of an upper bound.

Part B2: problems [45 points]

Question B2.1 [15 points]
(i) Determine if the statement below is true or false, and prove it.

For every odd \( k \in \mathbb{N} \), every \( a, b \in \mathbb{N} \), and every \( n \in \mathbb{N} \) such that \( n \geq 2 \), if \( a^k \equiv b^k \mod n \) then \( a \equiv b \mod n \).

(ii) Find a multiplicative inverse of 1, 2, 3, and 4 modulo 5.

Question B2.2 [15 points]
We define the relation \( R \) on \( \mathcal{P}(\mathbb{N}) \) via:

\[
\forall A, B \subseteq \mathbb{N}, ARB \iff A \setminus B \subseteq \{7\}
\]

Determine if it is reflexive, symmetric, transitive, and anti-symmetric. Prove your claims.

Question B2.3 [15 points]
Let \( S \) be a non-empty subset of the integers with an upper bound (with respect to the partial order \( \leq \) on \( \mathbb{Z} \)). Prove that \( S \) has a maximal element.
Part A1

Question A1.1 (see quiz 3.2)

Let $P(n)$, where $n \in \mathbb{N}$, be a mathematical proposition.

Suppose that:
(i) $P(0)$ holds
(ii) $\forall k \in \mathbb{N}, (\forall i \in \{0, \ldots, k\}, P(i)) \Rightarrow P(k+1)$

Then $P(n)$ holds for every $n \in \mathbb{N}$.

Question A1.2 (see quiz 4.1)

Every non-empty subset of the natural numbers has a least element.
Part A2  Question A2.1 (see lecture notes 3.1 and HW 4.1 for similar problems)

Base case: \((n = 1)\)

\[
4^{1+1} + 5^{2+1-1} = 4^2 + 5^1 = 16 + 5 = 21
\]
which is a multiple of 21, so the base case holds.

Induction step:

Let \(k \in \mathbb{N} \cap \{0\} \) and suppose that \(4^k + 5^{2k-1} \) is a multiple of 21, i.e. \(\exists a \in \mathbb{Z} \) such that

\[
4^k + 5^{2k-1} = 21a.
\]

Then

\[
4^{k+1} + 5^{2(k+1)-1} = 4^{k+1} + 5^{2k+1-1}
\]

\[
= 4 \cdot 4^k + 25 \cdot 5^{2k-1}
\]

\[
= 4 \cdot (4^k + 5^{2k-1}) + 21 \cdot 5^{2k-1}
\]

\[
= 4 \cdot 21a + 21 \cdot 5^{2k-1}
\]

by the induction hypothesis

\[
= 21(4a + 5^{2k-1})
\]

i.e. indeed \(4^{k+1} + 5^{2(k+1)-1} \) is a multiple of 21.

Question A2.2 (see additional problems on induction for a similar problem)

Base case: \((n = 1)\)

\[
\sum_{x=1}^{1} \frac{x-1}{x!} = 0 = 0 = \frac{1-1}{1} = \frac{1}{1} = 1
\]

Induction step:

Let \(k \in \mathbb{N} \cap \{0\} \) and suppose that \(\sum_{x=1}^{k} \frac{x-1}{x!} = \frac{x! - 1}{x!} \).

Then

\[
\sum_{x=1}^{k+1} \frac{x-1}{x!} = \left(\sum_{x=1}^{k} \frac{x-1}{x!}\right) + \frac{(k+1)-1}{(k+1)!}
\]

\[
= \frac{k! - 1}{k!} + \frac{k}{(k+1)!}
\]

by the induction hypothesis.
\[
\frac{(b+1)(b! - 1) + b}{(b+1)!} = \frac{(b+1)! - (b+1) + b}{(b+1)!} = \frac{(b+1)! - 1}{(b+1)!} \\
\text{i.e. indeed } \sum_{x=1}^{b+1} \frac{x-1}{x!} = \frac{(b+1)! - 1}{(b+1)!}.
\]

**Question A2.3** (see HW 4.1 for similar problems)

**Base case:** \((n = 0)\)

\[
\sum_{x=0}^{0} F_x^2 = F_0^2 = 0 = 0 \cdot 1 = F_0 F_1 \text{ so the base case holds.}
\]

**Induction step:**

Let \(k \in \mathbb{N} \) and suppose that \(\sum_{x=0}^{k} F_x^2 = F_k F_{k+1}\).

Then \(\sum_{x=0}^{k+1} F_x^2 = \left(\sum_{x=0}^{k} F_x^2\right) + F_{k+1}^2\)

\[
= F_k F_{k+1} + F_{k+1}^2 \quad \text{by the induction hypothesis}
\]

\[
= F_{k+1} (F_k + F_{k+1})
\]

\[
= F_{k+1} F_{k+2} \quad \text{by the definition of Fibonacci numbers}
\]

\[
\text{i.e. indeed } \sum_{x=0}^{k+1} F_x^2 = F_{k+1} F_{(k+1)+1}.
\]

**Question A2.4**

Suppose \(a_1 \) and \(a_2 \) are odd.

We prove by strong induction that \(a_n \) is odd for every \(n \in \mathbb{N} \).
Base cases: \( n = 1, 2 \)

\( a_1 \) and \( a_2 \) are odd by assumption.

Induction step:

Let \( n \in \mathbb{N} \) such that \( n \geq 2 \) and assume that for every \( i \in \{1, \ldots, k\} \), \( a_i \) is odd.

So for every \( i \in \{1, \ldots, k\} \), there exists \( b_i \in \mathbb{Z} \) such that \( a_i = 2b_i + 1 \).

Then \( a_{k+1} = 2a_k + 3a_{k-1} \) by definition of \( (a_n)_{n \in \mathbb{N}} \)

\[ a_{k+1} = 2(2b_{k-1} + 1) + 3(2b_{k-2} + 1) \]

\[ = 4b_{k-1} + 6b_{k-2} + 5 \]

\[ = 2(2b_{k-1} + 3b_{k-2} + 2) + 1 \]

i.e. indeed \( a_{k+1} \) is odd.
Part B1

Question B1.1
We say that \( x \) divides \( y \) if there exists \( a \in \mathbb{Z} \) such that \( y = ax \).

Question B1.2
Let \( A \) and \( B \) be sets. A relation between \( A \) and \( B \) is a subset of \( A \times B \).

Question B1.3 (see quiz 4.2)
Let \( A \) be a set and let \( R \) be an equivalence relation on \( A \). A subset \( C \) of \( A \) is called an equivalence class if \( C \) is non-empty and \( \forall x \in C, \forall y \in A, x R y \leftrightarrow y \in C \).

Question B1.4
A partial order is a relation which is reflexive, anti-symmetric, and transitive.

Question B1.5
Let \( A \) be a set, let \( R \) be a partial order on \( A \), let \( B \subseteq A \), and let \( x \in A \). We say that \( x \) is an upper bound of \( B \) if \( \forall y \in B, x R y \).
Part B2

Question B2.1 (see HW 4.2 for a problem similar to part (ii))

(i) The statement is false.

For example, if \( b = 3, a = 0, b = 2, \) and \( n = 4, \) we have that \( a^b \equiv b^a \mod n \) since
\[
\begin{align*}
a^b & \equiv b^a \mod 4 \\
0 & \equiv 0 \mod 4 \\
0 \cdot 2 & \equiv 2 \mod 4 \\
4 \cdot 2 & \equiv 2 \mod 4 \\
2^3 & \equiv 0 \mod 4 \\
l & \equiv l \mod 4
\end{align*}
\]

(ii) 1 is a multiplicative inverse of 1 modulo 5 since 1 \( \cdot \) 1 \( \equiv \) 1 \( \mod \) 5

\[
\begin{array}{ccccccccc}
3 & / & 2 & / & 3 & / & 2 & / & 3 \\
\hline \\
2 & / & 3 & / & 2 & / & 3 & / & 2
\end{array}
\]

Question B2.2 (see HW 4.2)

(a) Reflexivity: for any \( A \in \mathbb{N}, \) \( ARA \) since \( A \cdot A = A \leq \{0, \ldots, 7\} . \)

(ii) Failure of symmetry: we can take \( A = \emptyset \) and \( B = \{1, 2\} \) such that
\( A \cdot B = \emptyset \leq \{0, \ldots, 7\} \)

but \( B \cdot A = \{1, 2\} \neq \{0, \ldots, 7\} . \)

(iii) Transitivity: let \( A, B, C \subseteq \{0, \ldots, 7\} \) such that \( A \cdot B \) and \( B \cdot C, \)

i.e. \( A \cdot B \subseteq \{0, \ldots, 7\} \) and \( B \cdot C \subseteq \{0, \ldots, 7\} . \)

We want to show that \( A \cdot C \subseteq \{0, \ldots, 7\} . \)

Let \( x \in A \cdot C . \)

Case 1: \( x \in B \)

Then \( x \in B \cdot C, \) and since \( B \cdot C \) we know that \( B \cdot C \subseteq \{0, \ldots, 7\} \)

and hence \( x = 7, \) i.e. \( x \in \{0, \ldots, 7\} . \)

Case 2: \( x \notin B \)

Then \( x \in A \cdot B \subseteq \{0, \ldots, 7\} \) (since \( A \cdot B \)), i.e. \( x = 7, \)

such that \( x \in \{0, \ldots, 7\} . \)
Either way $x \in \{7\}$, i.e. indeed $A \subset \{7\}$.

(aiv) Failure of anti-symmetry: we can take $A = \emptyset$ and $B = \{7\}$. Then $A \setminus B = \emptyset \subseteq \{7\}$ and $B \setminus A = \{7\} \subseteq \{7\}$ but $A \neq B$.

**Question B2.3**

Let $S$ be a non-empty subset of $\mathbb{Z}$ with an upper bound. Since $S$ is non-empty, there exists $x \in S$.

Since $S$ has an upper bound, there exists $u \in \mathbb{Z}$ such that $\forall y \in S, y \leq u$.

Define $T = \{ z \in \mathbb{Z} \mid \exists s \in S \text{ such that } z = u - y \}$. Observe that $T$ is non-empty since $S$ is non-empty. In particular, $x \in S$ and hence $u - x \in T$.

Moreover, observe that $T \subseteq \mathbb{Z}$. Indeed, $\forall z \in T, \exists y \in S$ such that $z = u - y$ and since $u$ is an upper bound of $S$, we know that $y \leq u \Rightarrow u - y \geq 0$, i.e. $z \geq 0$ and indeed $z \in \mathbb{Z}$.

We can thus apply the well-ordering principle, which tells us that $T$ has a minimal element $\ell$.

Finally, we show that $u - \ell$ is a maximal element of $S$.

First, observe that since $\ell \in T$, $\exists y \in S$ such that $\ell = u - y$, i.e. $u - \ell = y \in S$, so indeed $u - \ell \in S$.

Second, observe that for every $y \in S$, $u - y \in T$ and hence $\ell < u - y$, i.e. $y < u - \ell$ and indeed $u - \ell$ is an upper bound of $S$.

Since $u - \ell$ is an element of $S$ and an upper bound of $S$, it is a maximal element of $S$. 
Part A: functions [140 points]

Part A1: definitions [50 points]

Question A1.1 [5 points]
Give the definition of left-totality.

Question A1.2 [5 points]
Give the definition of right-uniqueness.

Question A1.3 [5 points]
Give the definition of the image of a set under a function.

Question A1.4 [5 points]
Give the definition of a surjection.

Question A1.5 [5 points]
Give the definition of a bijection.

Question A1.6 [5 points]
Give the definition of a right-inverse.

Question A1.7 [5 points]
Give the definition of an inverse.

Question A1.8 [5 points]
Let $S$ and $T$ be sets. Give the definition of ‘$S$ has cardinality strictly larger than $T$’.

Question A1.9 [5 points]
Give the definition of an infinite set.

Question A1.10 [5 points]
Give the definition of a countably infinite set.
Part A2: problems [90 points]

Question A2.1 [15 points]
Prove or disprove the following claim:
Let $A$ and $B$ be sets, let $f : A \rightarrow B$ be a function, and let $X \subseteq A$. Then $X \subseteq \text{PreIm}_f (\text{Im}_f (X))$

Question A2.2 [15 points]
Let $f : A \rightarrow B$ be a function and let $S, T \subseteq A$. For each of the following claims, prove that it must hold, or disprove it by finding a counterexample.
(i) $\text{Im}_f (S \cup T) \subseteq \text{Im}_f (S) \cup \text{Im}_f (T)$
(ii) $\text{Im}_f (S) \cup \text{Im}_f (T) \subseteq \text{Im}_f (S \cup T)$

Question A2.3 [15 points]
A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called strictly increasing if for every $x, y \in \mathbb{R}, x < y \implies f(x) < f(y)$.
(i) Prove that any strictly increasing function is injective.
(ii) Prove that any strictly increasing function which is invertible has a strictly increasing inverse.

Question A2.4 [15 points]
Let $A, B,$ and $C$ be sets and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions such that $g$ is injective and $g \circ f$ is surjective. Prove that $f$ is surjective.

Question A2.5 [15 points]
Recall that for every $n \in \mathbb{N}$, $[n] = \{i \in \mathbb{N} | i < n\}$. Prove that for every nonzero $n \in \mathbb{N}$, $|\mathbb{N}| = |\mathbb{N} \times [n]|$.

Question A2.6 [15 points]
Let $A, B, C,$ and $D$ be sets such that $A \cong B$ and $C \cong D$. Prove that $A^C \cong B^D$. 
Part A1

Question A1.1
A relation $R$ between sets $A$ and $B$ is left-total if $\forall x \in A, \exists y \in B$ such that $x R y$.

Question A1.2
A relation $R$ between sets $A$ and $B$ is right-unique if $\forall x \in A, \forall y, z \in B$, $x R y \land x R z \Rightarrow y = z$.

Question A1.3
Let $f$ be a function from set $A$ to set $B$ and let $X$ be a subset of $A$.
The image of $X$ under $f$ is the set $\{ b \in B \mid \exists x \in X \text{ such that } f(x) = b \}$.

Question A1.4
A function $f : A \rightarrow B$ is a surjection if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.

Question A1.5
A bijection is a function which is both an injection and a surjection.

Question A1.6
Let $A$ and $B$ be sets and let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions.
We say that $g$ is a right-inverse of $f$ if $f \circ g = id_A$.

Question A1.7
Let $A$ and $B$ be sets and let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions.
We say that $g$ is an inverse of $f$ if $g$ is both a left-inverse and a right-inverse of $f$.

Question A1.8
$S$ has cardinality strictly larger than $T$ if $|S| > |T|$ and $|S| \neq |T|$.

Question A1.9
An infinite set is a set which is not finite.

Question A1.10
A set is countably infinite if it is isomorphic to $\mathbb{N}$. 
Part A2

Question A2.1
The claim is true.
Let \( a \in X \).
By definition of images, \( f(a) \in \text{Im} f(x) \).
By definition of pre-images: \( f(a) \in \text{Im} f(x) \iff a \in \text{Pre} \text{Im} f(x) \).
So indeed \( a \in \text{Pre} \text{Im} f(x) \).
Therefore, since \( a \in X \) was arbitrary, \( X \subseteq \text{Pre} \text{Im} f(x) \).

Question A2.2
We can show directly that \( \text{Im} f(S \cup T) = \text{Im} f(S) \cup \text{Im} f(T) \), i.e., we can show that both (i) & (ii) hold.
Let \( y \in B \). Then \( y \in \text{Im} f(S \cup T) \).
\[ \begin{align*}
&= \{ x \in S \cup T : f(x) = y \} \\
&= \{ x_1 \in S : f(x_1) = y \} \cup \{ x_2 \in T : f(x_2) = y \} \\
&= \{ x \in S : f(x) = y \} \cup \{ x \in T : f(x) = y \} \\
&= y \in \text{Im} f(S) \cup \text{Im} f(T)
\end{align*} \]
i.e., \( y \in \text{Im} f(S \cup T) \), \( y \in \text{Im} f(S) \cup \text{Im} f(T) \), and thus \( \text{Im} f(S \cup T) = \text{Im} f(S) \cup \text{Im} f(T) \).

Question A2.3
Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be strictly increasing.

(i) Let \( x, y \in \mathbb{R} \) such that \( x \neq y \).
Then either \( x < y \) or \( y < x \).
If \( x < y \), then \( f(x) < f(y) \), no in particular \( f(x) + f(y) \), and if \( y < x \), then \( f(y) < f(x) \), no again \( f(x) + f(y) \).
So either way: \( f(x) + f(y) \) & hence indeed \( f \) is injective.

(ii) Suppose \( f \) is invertible.
Let \( y_1, y_2 \in \mathbb{R} \) such that \( y_1 < y_2 \).
Suppose for the sake of contradiction that \( f^{-1}(y_1) \neq f^{-1}(y_2) \).
If \( f^{-1}(y_1) \neq f^{-1}(y_2) \), then \( y_1 = f(f^{-1}(y_1)) = f(f^{-1}(y_2)) = y_2 \), i.e. \( y_1 = y_2 \), which is a contradiction.
If \( f^{-1}(y_1) > f^{-1}(y_2) \), then, since \( f \) is strictly increasing, we obtain that \( f(y_1) > f(y_2) \), i.e. \( y_1 > y_2 \), a contradiction.
Either way we obtain a contradiction, which means that \( f^{-1}(y_1) = f^{-1}(y_2) \).

i.e. \( f^{-1} \) is strictly increasing.
**Question A2.4**

Let \( x \in B \).

We want to show that there exists \( x \in A \) such that \( f(x) = y \).

Observe that \( g(y) \in C \), therefore since \( g \circ f: A \to C \) is surjective, there exists \( x \in A \) such that \( (g \circ f)(x) = g(y) \), i.e. \( g(f(x)) = g(y) \).

Since \( g \) is injective, it follows from \( g(f(x)) = g(y) \) that \( f(x) = y \), so indeed we have found \( x \in A \) such that \( f(x) = y \), and indeed \( f \) is a surjection.

**Question A2.5**

Let \( n \in \mathbb{N} \setminus \{0\} \).

**Claim:**

(i) \( |n| \leq |n \times m| \)

(ii) \( |n \times m| \leq |n \times n| \)

Before we prove the claim, let us show that if the claim holds then \( |n| = |n \times m| \).

Suppose that the claim holds, and recall that \( |n| = |n \times n| \), i.e. \( |n| \leq |n \times n| \) and \( |n \times n| \leq |n| \).

Then:

\[ \{ \begin{array}{ll}
|n| \leq |n \times m| & \text{by part (i) of the claim} \\
|n \times m| \leq |n \times n| & \leq |n| & \text{by part (ii) of the claim}
\end{array} \]

Therefore, by Cantor–Schroeder–Bernstein, \( |n| = |n \times m| \).

**Proof of claim:**

(i) \( i: n \to n \times m \) given by \( i(n) = (n, 0) \) \( \forall n \in n \) is an injection, since \( \forall m, n \in n \), if \( (n, 0) = (m, 0) \), then \( m = n \).

(ii) \( j: n \times m \to n \times n \) given by \( j(q, r) = (q, r) \) \( \forall p \in n, q \in m \) is an injection.

**Question A2.6**

Since \( A \cong B \), there exists a bijection \( f: A \to B \) and since \( C \cong D \), there exists a bijection \( g: C \to D \).

Define \( H: A^C \to B^D \) by, \( \forall h \in A^C, H(h) = g \circ h \circ f^{-1} \) and define \( H: B^D \to A^C \) by, \( \forall h \in B^D, H(h) = f \circ h \circ g^{-1} \).

Let us show that \( H \) is a bijection. By showing that it is invertible.

We show that \( H \) is invertible by showing that \( H \circ H = \text{id}_{B^D} \) and \( H \circ H = \text{id}_{A^C} \).
\[ \forall \epsilon \in \mathbb{B}^D, (H \circ \overline{H})(\epsilon) = H \circ \overline{H}(\epsilon) \circ f^{-1} = \overline{f} \circ \overline{H}(\epsilon) \circ f \circ f^{-1} = \overline{f} \]

\[ \forall \epsilon \in \mathbb{A}^C, (H \circ \overline{H})(\epsilon) = H \circ \overline{H}(\epsilon) \circ f^{-1} = \overline{f} \circ \overline{H}(\epsilon) \circ f \circ f^{-1} = \overline{f} \]

i.e. indeed:

\[ H \circ \overline{H} = \text{id}_{\mathbb{B}^D} \]

\[ \overline{H} \circ H = \text{id}_{\mathbb{A}^C} \]