Sets and subsets

Question 1 [5 points]

Give the definition of a subset.

Let $A$ and $B$ be sets. We say that $A$ is a subset of $B$ if every element of $A$ is also an element of $B$.

Question 2 [5 points]

Give the definition of set equality.

Let $A$ and $B$ be sets. We say that $A$ and $B$ are equal if $A \subseteq B$ and $B \subseteq A$. 
Set operations

Question 1 [5 points]
Give the definition of the union of two sets.

Let A and B be sets. The union of A and B is the set of all elements which belong to either A or B.

Question 2 [5 points]
Give the definition of set difference.

Let A and B be sets. The difference between A and B is the set of all elements of A which are not elements of B.
Indexed sets

Question 3 [5 points]
Give the definition of an indexed intersection of sets.

Let \( I \) be a set and let \( \{A_i\}_{i \in I} \) be a collection of sets.

Then \( \bigcap_{i \in I} A_i = \{ x | x \in A_i \text{ for all } i \in I \} \).

Question 4 [5 points]
Give the definition of a partition.

Let \( S \) be a set. A partition of \( S \) is a collection of pairwise disjoint subsets of \( S \) covering \( S \).
Cartesian products

Question 1 [5 points]
Give the definition of the Cartesian product of two sets.

Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$ is the set of ordered pairs $(a, b)$ where $a \in A$ and $b \in B$.

Question 2 [5 points]
Find two sets $A$ and $B$ such that $A \times B \neq B \times A$.

$A = \{1\}, \quad B = \{2\}$
Mathematical statements

Question 3 [5 points]
Give the definition of a mathematical statement.

A mathematical statement is a grammatically correct sentence, composed of English words and mathematical symbols, that has exactly one truth value (either true or false).

Question 4 [5 points]
Give the definition of a mathematical proposition.

A mathematical proposition is a grammatically correct sentence, composed of English words and mathematical symbols, and depending on one or more variables, such that for every value(s) of the variable(s), the sentence has exactly one truth value.
Logical connectives

Question 1 [5 points]
Let $P$ and $Q$ be mathematical statements. Give the definition of $P \land Q$.

"$P \land Q$ is true when both $P$ and $Q$ are true, and is false otherwise."

Question 2 [5 points]
Let $P$ and $Q$ be mathematical statements. Give the definition of $P \implies Q$.

"$P \implies Q$ is false when $P$ is true and $Q$ is false, and is true otherwise."
Negation

Question 3 [5 points]

Give the definition of logical negation.

Let $P$ be a mathematical statement or proposition. The negation of $P$ has truth values opposite to the truth values of $P$.

Question 4 [5 points]

For each of the sets in (i) - (iv), match it with the corresponding set in (a) - (d).

(i) $\left( \bigcap_{i \in I} \bigcup_{j \in J} S_{i,j} \right)^c$  
(ii) $\left( \bigcap_{i \in I} \bigcap_{j \in J} S_{i,j} \right)^c$  
(iii) $\left( \bigcup_{i \in I} \bigcap_{j \in J} S_{i,j} \right)^c$  
(iv) $\left( \bigcup_{i \in I} \bigcup_{j \in J} S_{i,j} \right)^c$

(a) $\bigcap_{i \in I} \bigcap_{j \in J} S_{i,j}^c$  
(b) $\bigcap_{i \in I} \bigcup_{j \in J} S_{i,j}^c$  
(c) $\bigcup_{i \in I} \bigcap_{j \in J} S_{i,j}^c$  
(d) $\bigcup_{i \in I} \bigcup_{j \in J} S_{i,j}^c$  
(e) - (c)  
(f) - (d)  
(iii) - (b)  
(iii) - (a)
Logical equivalence

Question 1 [5 points]
Give the definition of logical equivalence.

Let $P$ and $Q$ be mathematical statements or propositions. We say that $P$ and $Q$ are equivalent when they have the same truth values.

Question 2 [5 points]
Let $P$ and $Q$ be mathematical statements. Show that $Q \Rightarrow (\neg P)$ is equivalent to $\neg (P \land Q)$.

$$Q \Rightarrow (\neg P) \iff (\neg Q) \lor (\neg P)$$

$$\iff \neg (P \land Q)$$
Induction

Question 3 [5 points]
State the principle of mathematical induction.

Let \( P(n) \), where \( n \in \mathbb{N} \), be a mathematical proposition.

(i) Suppose that \( P(0) \) holds

(ii) \( \forall n \in \mathbb{N}, \ (P(n) \Rightarrow P(n+1)) \) holds

Then \( P(n) \) holds for every \( n \in \mathbb{N} \).

Question 4 [5 points]
Let \( x \in \mathbb{R} \) be such that \( x \neq 1 \). Prove that

\[
\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}
\]

for every \( n \in \mathbb{N} \).

Note that the indexed sum is defined such that \( \sum_{i=0}^{-1} a_i = 0 \) for any sequence of real numbers \( (a_i) \). For reference: such a sum is called an **empty sum**.

**Base case:** \( \sum_{i=0}^{0} x^i = 0 \) by definition.

**Induction step:** Let \( k \in \mathbb{N} \) and suppose that \( \sum_{i=0}^{k-1} x^i = \frac{x^{k-1}}{x - 1} \).

Then:

\[
\sum_{i=0}^{k} x^i = (\sum_{i=0}^{k-1} x^i) + x^k = \frac{x^{k-1}}{x - 1} + x^k \quad \text{by the induction hypothesis}
\]

\[
= \frac{x^{k} - 1 + x^{k+1} - x^k}{x - 1} = \frac{x^{k+1} - 1}{x - 1}
\]
Variants of induction

Question 1 [5 points]

Let $m$ be a natural number. Find the flaw in the statement below and change one symbol to correct it.

If $T$ is a set of natural numbers such that

(i) $m \in T$

(ii) $n \in T$ implies $n + 1 \in T$

then $\{n \in \mathbb{N} \mid n \geq m\} \subseteq T$.

Should be $\subseteq$.

Question 2 [5 points]

For which $n \in \mathbb{N}$ is $2^n > n^2 + 2$? Prove your claim.

Claim: $2^n > n^2 + 2 \iff n \geq 5$

Proof: For $n \leq 4$, $2^n < n^2 + 2$ as shown in the table.

For $n > 5$, we proceed by induction:

Base case: ($n = 5$) $2^5 = 32 > 27 = 5^2 + 2$

Induction step: Let $k \in \mathbb{N}$ such that $k \geq 5$ and $2^k > k^2 + 2$.

Then $2^{k+1} = 2 \cdot 2^k > 2(k^2 + 2)$ by the induction hypothesis

$= 2k^2 + 4$

$= (k^2 + 2k + 3) + (k^2 - 2k + 1)$

$= (k+1)^2 + 2 \cdot (k-1)^2$

$> (k+1)^2 + 2$

PLEASE TURN OVER
Strong induction

Question 3 [5 points]
State the principle of strong mathematical induction.

Let $P(n)$, where $n \in \mathbb{N}$, be a mathematical proposition.

Suppose that:
(i) $P(0)$ holds,
(ii) $\forall k \in \mathbb{N}, \ (\forall j \in \{1, \ldots, k\}, \ P(j)) \Rightarrow P(k+1)$

Then $P(n)$ holds for every $n \in \mathbb{N}$.

Question 4 [5 points]
The Fibonacci numbers $(F_n)_{n \in \mathbb{N}}$ are defined via:

$$
\begin{cases}
F_0 = 0 \\
F_1 = 1 \\
F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2
\end{cases}
$$

Prove that

$$F_n F_{n+2} = F_{n+1}^2 + (-1)^{n+1}$$

for all $n \in \mathbb{N}$.

**Base case**: $n = 0$

$$
\begin{align*}
F_0 F_2 &= 1 \cdot 0 = 0 \\
F_1^2 + (-1)^1 &= 1 - 1 = 0
\end{align*}
$$

**Induction step**: Let $k \in \mathbb{N}$ such that $F_{k+1} F_{k+2} = F_{k+2}^2 + (-1)^{k+1}$.

Then $F_{k+1} F_{k+3} = F_{k+2} F_{k+4}$

$$= (F_{k+2} + F_{k+3}) F_{k+4} \quad \text{by the definition of Fibonacci numbers}$$

$$= F_{k+2} F_{k+4} + F_{k+3}^2$$

$$= F_{k+2} F_{k+4} + (F_{k+2} F_{k} - (-1)^{k+2}) \quad \text{by the induction hypothesis}$$

$$= F_{k+2} (F_{k+1} + F_{k}) + (-1)^{k+2}$$

$$= F_{k+2}^2 + (-1)^{k+2} \quad \text{by the definition of Fibonacci numbers}$$
Variants of strong induction

Question 1 [5 points]
State the well-ordering principle.

Every non-empty subset of the natural numbers has a least element.

Question 2 [5 points]
Prove that for all nonzero \( n \in \mathbb{N} \),

\[
\sum_{i=n}^{2n-1} (2i+1) = 3n^2
\]

Note that the summation index \( i \) ranges from \( n \) to \( 2n - 1 \).

Base case: \( (n=1) \)

\[
\sum_{i=1}^{2(1)+1} (2i+1) = 3 = 3 \cdot 1^2
\]

Induction step: Let \( k \in \mathbb{N} \) and assume that \( \sum_{i=k}^{2k-1} (2i+1) = 3k^2 \).

Then

\[
\sum_{i=k}^{2(k+1)-1} (2i+1) = \sum_{i=k+1}^{2k+1} (2i+1)
\]

\[
= \left( \sum_{i=k}^{2k-1} (2i+1) \right) - (2k+1) + (2(2k)+1) + (2(2k+1)+1)
\]

\[
= 3k^2 - (2k+1) + (4k+1) + (4k+3)
\]

\[
= 3k^2 + 6k + 3
\]

\[
= 3(k+1)^2
\]
Modular arithmetic

Question 3 [5 points]

Give the definition of congruence.

let \( n \in \mathbb{N} \) and let \( x, y \in \mathbb{Z} \). We say that \( x \) and \( y \) are congruent modulo \( n \) if \( n \) divides \( x - y \).

Question 4 [5 points]

What is the last digit of \( 3447^6 + (8545 - 98263)^{15} \)?

\[
3447^6 + (8545 - 98263)^{15} \equiv 7^6 + (5 - 3)^{15} \mod 10 \\
\equiv 49^3 + 2^{15} \mod 10 \\
\equiv 9^3 + (2^5)^3 \mod 10 \\
\equiv 81 + 32^3 \mod 10 \\
\equiv 1 + 2^3 \mod 10 \\
\equiv 9 + 8 \mod 10 \\
\equiv 17 \mod 10 \\
\equiv 7 \mod 10
\]

Therefore, the last digit of \( 3447^6 + (8545 - 98263)^{15} \) is 7.
Relations

Question 1 [5 points]
Let $R$ be a relation on a set $A$. Give the definition of $R$ being symmetric.

\[ \forall x, y \in A, \quad x R y \Rightarrow y R x \]

Question 2 [5 points]
Let $R$ be a relation on a set $A$. Give the definition of $R$ being anti-symmetric.

\[ \forall x, y \in A, \quad (x R y \land y R x) \Rightarrow x = y \]
Equivalence relations

Question 3 [5 points]
Give the definition of an equivalence relation.

An equivalence relation is a relation which is reflexive, symmetric, and transitive.

Question 4 [5 points]
Give the definition of an equivalence class.

Let $A$ be a set and let $R$ be an equivalence relation on $A$. A subset $C$ of $A$ is called an equivalence class of $R$ if $C$ is non-empty and $\forall x \in C, \forall y \in A, \ x R y \Leftrightarrow y \in C$. 
Order relations

Question 1 [5 points]
Give the definition of a total order.

A total order is a partial order with respect to which every pair of elements is comparable.

Question 2 [5 points]
Give the definition of a minimal element.

Let $A$ be a set, let $\mathcal{R}$ be a partial order on $A$, let $B \subseteq A$, and let $x \in A$. We say that $x$ is a minimal element of $B$ if $x \in B$ and $\forall y \in B, x \mathcal{R} y$. 
Functions

Question 3 [5 points]
Give the definition of right-uniqueness.

A relation $R$ between a set $A$ and a set $B$ is right-unique if $\forall x \in A, \forall y, z \in B, (xRy \land yRx) \Rightarrow y = z$.

Question 4 [5 points]
Give the definition of a function.

A function is a relation which is left-total and right-unique.
Images and pre-images

Question 5 [5 points]
Let $A$, $B$ be sets, let $f : A \to B$ be a function, and let $S \subseteq A$. Give the definition of the image of $S$ under $f$.

The image of $S$ under $f$ is $\text{Im}(f)(S) = \{y \in B \mid \exists x \in A \cdot f(x) = y\}$. 
Images and pre-images

Question 1 [5 points]

Give the definition of the pre-image of a set under a function.

Let $A$ and $B$ be sets, let $f: A \rightarrow B$ be a function, and let $Y \subseteq B$. The pre-image of $Y$ under $f$ is $\{ x \in A | f(x) \in Y \}$.
Properties of functions

Question 2 [5 points]
Give the definition of an injection.

Let $A$ and $B$ be sets and let $f: A \rightarrow B$ be a function. We say that $f$ is an injection if $\forall x, y \in A, \ x \neq y \Rightarrow f(x) \neq f(y)$.

Question 3 [5 points]
Give the definition of a surjection.

Let $A$ and $B$ be sets and let $f: A \rightarrow B$ be a function. We say that $f$ is a surjection if $\forall y \in B, \exists x \in A$ such that $y = f(x)$. 
Compositions and inverses

Question 1 [5 points]

Give the definition of the composition of two functions.

Let $A$, $B$, and $C$ be sets and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. The composition of $g$ with $f$, denoted $g \circ f$, is a function $g \circ f : A \rightarrow C$ such that for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$
Composition and inverses

Question 1 [5 points]
Give the definition of a right-inverse.

Let A and B be sets and let \( f: A \rightarrow B \) and \( g: B \rightarrow A \) be functions. We say that \( g \) is a right-inverse of \( f \) if \( f \circ g = \text{id}_B \).

Question 2 [5 points]
Give the definition of an inverse.

Let A and B be sets and let \( f: A \rightarrow B \) and \( g: B \rightarrow A \) be functions. We say that \( g \) is an inverse of \( f \) if \( f \circ g = \text{id}_B \) and \( g \circ f = \text{id}_A \).
Cardinality

Question 3 [5 points]
Give the definition of 'countably infinite'.

A set $S$ is countably infinite if $S \cong \mathbb{N}$.

Question 4 [5 points]
Let $S$ and $T$ be sets. Give the definition of 'S has cardinality at least $|T|$'.

$S$ has cardinality at least $|T|$ if there exists a surjection from $S$ to $T$. 