5.5. Cardinality

Notation
For any $m \in \mathbb{N}$, we define $[m] = \{i \in \mathbb{N} \mid 0 \leq i < m\} = \{0, 1, \ldots, m-1\}$,
e.g.: $[0] = \emptyset$
$[1] = \{0\}$
$[2] = \{0, 1\}$
Crucially: $[m]$ has $m$ elements.

Definition (isomorphic sets)
Let $A$ and $B$ be sets. If there exists a bijection $f : A \to B$, then we say that $A$ and $B$ are isomorphic, and we write $A \cong B$.

Definition (finite, infinite, countably infinite)
Let $S$ be a set.
(i) If there exists $m \in \mathbb{N}$ such that $S \cong [m]$, then we say that $S$ has size (or cardinality) $m$, and we write $|S| = m$.
(ii) If $S$ is not finite, then we say that $S$ is infinite.
(iii) If $S \cong [\mathbb{N}]$, then we say that $S$ is countably infinite.

Examples
(1) $\forall m, n \in \mathbb{N}, \quad [m] \cong [n] \cong [k]$, i.e. $|[m]| = n$
N.B. Elements of $\mathbb{N}^{\mathbb{N}}$ look like sequences of $n$ numbers between 0 and $n-1$.

(2) Let $E = \{\text{even natural numbers}\}$ and $O = \{\text{odd natural numbers}\}$.
Then $\mathbb{N} \cong E \cong O$ since
\[
\begin{align*}
f : \mathbb{N} &\to E, f(n) = 2n \quad \forall n \in \mathbb{N} \\
g : \mathbb{N} &\to O, g(n) = 2n+1 \quad \forall n \in \mathbb{N}
\end{align*}
\]
are bijections.
Hence $|\mathbb{N}| = |E| = |O| = 101$.

Definition (comparing cardinalities)
Let $S$ and $T$ be sets.
(i) We say $S$ has the same cardinality as $T$, and write $|S| = |T|$, if there exists a bijection $f : S \to T$ (i.e. if $S \cong T$).
(ii) We say $S$ has cardinality at most $|T|$, and write $|S| \leq |T|$, if there exists an injection $f : S \to T$.
(iii) We say $S$ has a strictly smaller cardinality than $T$, and write $|S| < |T|$, if $|S| \leq |T|$ and $|S| \neq |T|$.
(iv) We say \( S \) has cardinality at least \( |T| \), and write \( |S| \geq |T| \), if there exists a surjection \( f : S \to T \).

(v) We say \( S \) has a strictly larger cardinality than \( T \), and write \( |S| > |T| \), if \( |S| \geq |T| \) and \( |S| \neq |T| \).

**Remark**

Do not be fooled by this suggestive notation which makes some statements look simpler than they are. For example, consider:

1. For any sets \( A \) and \( B \), \( |A| < |B| \iff |B| > |A| \)
2. For any sets \( A \) and \( B \), \( |A| < |B| \vee |B| < |A| \)
3. For any sets \( A \) and \( B \), \( |A| < |B| \wedge |B| < |A| \) \( \implies |A| = |B| \)

One of these statements is easy to prove (and we will), one is difficult to prove (and we will not), and one is axiom!

**Lemma**

Let \( A \) and \( B \) be sets. Then \( |A| \leq |B| \iff |B| \geq |A| \),

i.e. there is an injection from \( A \) to \( B \) if and only if there is a surjection from \( B \) to \( A \).

**Proof**

\[
|A| < |B| \iff \exists \text{ injection } f : A \to B \\
\iff \exists \text{ left-invertible } f : A \to B \\
\iff \exists f : A \to B, \exists g : B \to A \text{ s.t. } g \circ f = \text{id}_A \\
\iff \exists \text{ right-invertible } g : B \to A \\
\iff \exists \text{ bijection } g : B \to A \\
\iff |B| \geq |A|
\]

**Axiom (Principle of Cardinal Comparability)**

Given any two sets \( A \) and \( B \), we must have \(|A| \leq |B| \) or \(|B| \leq |A| \) (or both),

i.e. there must be an injection from \( A \) to \( B \) or an injection from \( B \) to \( A \).

**Theorem (Cantor-Schröder-Bernstein)**

Given any two sets \( A \) and \( B \), \( \exists f : A \to B, \exists g : B \to A \) \( \iff |A| = |B| \),

i.e. if there exist injections from \( A \) to \( B \) and from \( B \) to \( A \), then there exists a bijection between \( A \) and \( B \).
Remark
Consider the "relation" $R$ defined via: for any two sets $A$ and $B$, $A R B \iff |A| \leq |B|$. Then:
1. The Principle of Cardinal Comparability says that every pair of sets is $R$-comparable.
2. The Cantor-Schröder-Bernstein Theorem says that $R$ is anti-symmetric.

Corollary
For any two sets $A$ and $B$, $\neg (|A| > |B|) \iff |A| < |B|$

Proof
$\Leftarrow$
Suppose $\neg (|A| > |B|)$, i.e. $\neg (|B| \leq |A|)$ (by the lemma above).
Then, by the Principle of Cardinal Comparability we must have $|A| \leq |B|$.
So it is enough to show that $|A| \leq |B|$. 
Suppose for the sake of contradiction that $|A| = |B|$.
Then in particular $|A| \leq |B|$, a contradiction.
So indeed $|A| < |B|$ and $|A| \neq |B|$, i.e. $|A| < |B|$.

$\Rightarrow$
Suppose $|A| < |B|$, i.e. $|A| < |B|$ and $|A| < |B|$.
Suppose for the sake of contradiction that $|A| > |B|$.
Then, by the lemma above, $|B| < |A|$.
So by anti-symmetry, i.e., by Cantor-Schröder-Bernstein, $|A| = |B|$.
This is a contradiction, so indeed $\neg (|A| > |B|)$.

Lemma
For any three sets $A$, $B$, and $C$,
(i) If $|A| < |B|$ and $|B| < |C|$ then $|A| < |C|$
(ii) If $|A| < |B|$ and $|B| < |C|$ then $|A| < |C|$

Proof
(i) Let $A$, $B$, and $C$ be sets such that $|A| < |B|$ and $|B| < |C|$.
This means there exist injections $f: A \rightarrow B$ and $g: B \rightarrow C$.
Therefore, by a previous result, $g \circ f: A \rightarrow C$ is an injection and indeed $|A| < |C|$. 

(iii) Let $A, B,$ and $C$ be sets such that $|A| \leq |B|$ and $|B| \leq |C|$. In particular $|A| \leq |B|$ and $|B| \leq |C|$, and hence, by part (ii), $|A| \leq |C|$.

We now want to show that $|A| \neq |C|$.

Suppose for the sake of contradiction that $|A| = |C|$.

Since $|C| \leq |A|$ and $|A| \leq |B|$, it follows from part (ii) that $|C| \leq |B|$.

In particular $|C| = |B|$, which contradicts $|B| < |C|$.

Thus $|A| \neq |C|$, and hence indeed $|A| < |C|$.

**Theorem**

Let $S$ be a set. Then $|S| < |\mathcal{P}(S)|$.

**Proof**

By the corollary above: $|S| < |\mathcal{P}(S)| \Rightarrow \neg (|S| \geq |\mathcal{P}(S)|)$.

Suppose, for the sake of contradiction, that there exists a surjection $f: S \rightarrow \mathcal{P}(S)$.

Define $T = \{x \in S | x \notin f(x)\}$.

Since $T \subseteq \mathcal{P}(S)$ and since $f$ is a surjection, there exists $x_0 \in S$ such that $f(x_0) = T$.

Now observe that: $x_0 \in T \iff x_0 \notin f(x_0) \iff x_0 \notin T$

by definition of $T$, since $f(x_0) = T$

which is a contradiction.
Proposition

$|\mathbb{Z}| = |\mathbb{N}|$, i.e. $\mathbb{Z}$ is countable

Proof

Consider $f: \mathbb{Z} \to \mathbb{N}$ given by, $\forall z \in \mathbb{Z}$, $f(z) = \begin{cases} -2z & \text{if } z < 0 \\ 2z-1 & \text{if } z > 0 \end{cases}$

and $g: \mathbb{N} \to \mathbb{Z}$ given by, $\forall n \in \mathbb{N}$, $g(n) = \begin{cases} \frac{1}{2}(n+1) & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$

Then $g \circ f = \text{id}_{\mathbb{Z}}$, so $f$ and $g$ are invertible, hence bijective, which means that $\mathbb{Z} \cong \mathbb{N}$.

Proposition

$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, i.e. $\mathbb{N} \times \mathbb{N}$ is countable

Proof

We will use the following claim:

Claim: $\forall n \in \mathbb{N}$, $f(1, n) = \frac{n}{2}$ if $n$ is even.

Proof of claim: Exercise (use induction).

The claim says that $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.

Now consider $g: \mathbb{N} \to \mathbb{N}$ given by $g(n) = n - 1 \forall n \in \mathbb{N}$.

Observe that $g$ is a bijection.
So finally, \( \circ f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is a composition of bijections, hence a bijection.

**Lemma**

Let \( A, B \) be sets.

If \( A \) and \( B \) are countably infinite, then \( A \times B \) is countably infinite.

**Proof**

We want to show that \( A \times B \cong \mathbb{N} \).

Since the proposition above tells us that \( \mathbb{N} \times \mathbb{N} \cong \mathbb{N} \), it is enough to show that \( A \times B \cong \mathbb{N} \times \mathbb{N} \).

Note that since \( A \) and \( B \) are countably infinite, there exist bijections \( f : A \to \mathbb{N} \) and \( g : B \to \mathbb{N} \).

Define \( H : A \times B \to \mathbb{N} \times \mathbb{N} \) by, for all \( a \in A, b \in B \), \( H(a, b) = (f(a), g(b)) \), and define \( \overline{H} : \mathbb{N} \times \mathbb{N} \to A \times B \) by, for all \( p, q \in \mathbb{N} \), \( \overline{H}(p, q) = (f^{-1}(p), g^{-1}(q)) \).

Let us show that

\[
\begin{align*}
H \circ \overline{H} &= \text{id}_{\mathbb{N} \times \mathbb{N}} \quad (1) \\
\overline{H} \circ H &= \text{id}_{A \times B} \quad (2)
\end{align*}
\]
(1) \( \forall p, q \in \mathbb{N}, \ H(f(p), g(q)) = H(f^{-1}(p), g^{-1}(q)) \)
\[
= \left( f(f^{-1}(p)), g(g^{-1}(q)) \right)
\]
\[
= (p, q)
\]
\[
= \text{id}_{\mathbb{N} \times \mathbb{N}}(p, q)
\]

(2) \( \forall a \in A, b \in B, \ H(H(a, b)) = H(f(a), g(b)) \)
\[
= \left( f^{-1}(f(a)), g^{-1}(g(b)) \right)
\]
\[
= (a, b)
\]

Since \( H \) & \( F \) are invertible, they are injections, so indeed \( A \times B \cong \mathbb{N} \times \mathbb{N} \).

**Theorem**

\( \mathbb{Q} \) is countably infinite.

**Proof**

We want to show that \( |\mathbb{Q}| = |\mathbb{N}| \).

Clearly \( |\mathbb{N}| \leq |\mathbb{Q}| \) since \( \iota: \mathbb{N} \to \mathbb{Q} \) where \( \iota(n) = \frac{n}{n} \), is an injection.

Now let us show that \( |\mathbb{N}| \geq |\mathbb{Q}| \).

By the results above, \( |\mathbb{Z}| = |\mathbb{N}| \).

\[
|\mathbb{Z} \times \mathbb{N} : 0 \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|
\]

Since both \( \mathbb{Z} \) & \( \mathbb{N} \times \mathbb{N} \) are countably infinite, so it is enough to show that \( |\mathbb{Z} \times \mathbb{N} : 0 \times \mathbb{N}| \). This is immediate since \( \iota: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q} \) where \( \iota(g, m) = \frac{g}{m} \) is a surjection.
Definition (Uncountable set)
We say a set is uncountable if it is infinite but not countably infinite.

Theorem
\( \mathbb{R} \) is uncountable.

Lemma
\((0,1)\) is isomorphic to \( \mathbb{R} \).

Proof (of lemma)
Consider \( f : (0,1) \to \mathbb{R} \) defined by, \( \forall x \in (0,1), \ f(x) = \begin{cases} \frac{1}{x} - 2 & \text{if } x < \frac{1}{2} \\ 2 - \frac{1}{1-x} & \text{if } x > \frac{1}{2} \end{cases} \)

\( f \) is a bijection since it has inverse \( g : \mathbb{R} \to (0,1) \) given by, \( \forall y \in \mathbb{R}, \ g(y) = \begin{cases} \frac{1}{y+2} & \text{if } y \geq 0 \\ 1 + \frac{1}{y-2} & \text{if } y < 0 \end{cases} \)
Proof (of theorem)

We want to show that \(|\mathbb{N}| < |\mathbb{R}|\), which is equivalent to \(-(|\mathbb{N}| \geq |\mathbb{R}|)\), i.e., we want to show that there is no surjection from \(\mathbb{N}\) to \(\mathbb{R}\).

Since \(\mathbb{R}\) is isomorphic to \((0,1)\), it is enough to show that there is no surjection from \(\mathbb{N}\) to \((0,1)\).

Let \(f: \mathbb{N} \rightarrow (0,1)\) be any function.

Every \(x \in (0,1)\) has a unique decimal representation, so let \(a_j\) be the \((j+1)\)-th decimal of \(f(x)\), i.e.,

\[
\begin{align*}
f(0) &= 0.a_{00}a_{01}a_{02}a_{03} \ldots \\
f(1) &= 0.a_{10}a_{11}a_{12}a_{13} \ldots \\
f(2) &= 0.a_{20}a_{21}a_{22}a_{23} \ldots \\
&\text{etc.}
\end{align*}
\]

For each \(b \in \mathbb{N}\), let \(b_f = \begin{cases} a_{bb} - 1 & \text{if } 1 \leq ab_b < 9 \\ 9 & \text{if } ab_b = 0 \end{cases}\)

\[
0 \rightarrow 0 \\
1 \rightarrow 1 \\
2 \rightarrow 2 \\
8 \rightarrow 8 \\
9 \rightarrow 9
\]

and let \(x^* = 0.b_1b_2b_3 \ldots\).

Note that \(\{x \in (0,1) : x \neq f(b) \text{ for all } b \in \mathbb{N}\}\) i.e., \(x \notin \text{Inf}(\mathbb{N})\).
In other words, if cannot be surjective.