Definition (composition of functions)
Let $A$, $B$, and $C$ be sets, let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. We define the composition of $g$ with $f$, denoted $g \circ f$, and sometimes read "$g$ after $f"$, as the function $g \circ f: A \rightarrow C$ given by $\forall x \in A, (g \circ f)(x) = g(f(x))$. 

Examples
(i) 

\[ 
\begin{align*} 
\text{A} & \quad \text{B} & \quad \text{C} \\
1 & \rightarrow a & \rightarrow \text{France} \\
2 & \rightarrow b & \rightarrow \text{Croatia} \\
3 & \rightarrow c & \rightarrow \text{Brazil} \\
4 & \rightarrow e & \rightarrow \text{France} \\
\end{align*} 
\]
(2) Consider \( C, F : \mathbb{R} \to \mathbb{R} \), where \( V \in \mathbb{R} \)

\[
C(x) = x - 273.15
\]

\[
F(x) = \frac{9}{5}x + 32
\]

i.e., \( \text{Kelvin} \rightarrow \text{Celsius} \rightarrow \text{Fahrenheit} \)

\[
(F \circ C)(x) = F(C(x))
= \frac{9}{5}C(x) + 32
= \frac{9}{5}(x - 273.15) + 32
= \frac{9}{5}x - 459.67
\]

(3) For any function \( f : A \to B \), we have that \( f \circ id_A = f = id_B \circ f \).

Remarks

(1) Function composition is associative: for any \( f : A \to B \), \( g : B \to C \), and \( h : C \to D \), we have that \( h \circ (g \circ f) = (h \circ g) \circ f \), so we omit the parentheses and write \( h \circ g \circ f \).

(2) Any function \( f : A \to B \) induces functions \( \text{Inf} : \mathcal{P}(A) \to \mathcal{P}(B) \) and \( \text{Inf}^{-1} : \mathcal{P}(B) \to \mathcal{P}(A) \).

Proposition (composition \& injectivity)

Let \( A, B, \) and \( C \) be sets, let \( f : A \to B \) and \( g : B \to C \) be functions.

(i) If \( f \) and \( g \) are injections, then \( g \circ f \) is an injection.

(ii) If \( g \circ f \) is an injection, then \( f \) is an injection.

Proof (see next page for (ii))

(ii) Suppose \( g \circ f \) is an injection.

We want to show that \( \forall x, y \in A \), \( x \neq y \Rightarrow f(x) \neq f(y) \)

\[\Leftrightarrow \quad (f(x) = f(y) \Rightarrow x = y)\]

So let \( x, y \in A \) and suppose \( f(x) = f(y) \).

Applying \( g \) to both sides yields \( g(f(x)) = g(f(y)) \),
**Proof of (ii):** Let \( f, g \) be injections and let \( x, y \in A \) such that \( x \neq y \). Then \( f(x) \neq f(y) \) since \( f \) is injective, so \( g(f(x)) \neq g(f(y)) \) since \( g \) is injective, and so indeed \( (g \circ f)(x) \neq (g \circ f)(y) \), i.e. \( g \circ f \) is injective.

Therefore, since \( g \circ f \) is injective, it follows that \( x = y \).

We have thus shown that \( f(x) = f(y) \Rightarrow x = y \), i.e. that \( f \) is injective.

**Remark:**

\( g \circ f \) is injective \( \neq \) \( g \) is injective.

Consider, for example \( A = \{1, 2\} \)
\( B = \{a, b, \text{potato}\} \)
\( f(1) = a \)
\( g(a) = g(b) = \text{potato} \)
\( g \) is not an injection.

Then \( (g \circ f)(1) = g(f(1)) = g(a) = \text{potato} \), and \( g \circ f \) is an injection.

![Diagram](attachment:image.png)

**Definition (right-inverse):**

Let \( A \) and \( B \) be sets, and let \( f: A \rightarrow B \) be a function. If there exists a function \( g: B \rightarrow A \) such that \( f \circ g = \text{id}_B \), then we say that \( g \) is a right-inverse of \( f \), and we say that \( f \) is right-invertible.

**Proposition (surjectivity \( \iff \) right-invertibility):**

Let \( A, B \) be sets, and let \( f: A \rightarrow B \) be a function.

Then: \( f \) is surjective \( \iff \) \( f \) is right-invertible.

**Proof:**

\( \Leftarrow \) Suppose \( f \) is right-invertible. Then \( f \) has a right-inverse, let us call it \( g \). Since \( g \) is a right-inverse of \( f \),

\[ \forall x \in B, \quad g(f(x)) = x \]

Then \( f(x) = f(g(f(x))) = (f \circ g)(x) = x \)

i.e. \( f \) is surjective.

\( \Rightarrow \) Suppose \( f \) is surjective. Let \( g \) be any function such that \( f \circ g = \text{id}_B \).

Now: \( \forall x \in B, \quad f(g(x)) = x \)

Then \( f(x) = f(g(f(x))) = (f \circ g)(x) \)

i.e. \( f \) is right-invertible.
Suppose \( f : A \rightarrow B \) is surjective.
Then \( \{ \text{pre} f (x) \mid x \in B \} \) is a collection of non-empty subsets of \( A \) so there exists \( g : B \rightarrow A \) such that \( \forall x \in B, \ g (y) \in \text{pre} f (x) \), i.e. \( f (g (y)) = y \), or in other words: \( g \) is a right-inverse for \( f \).

Remark/Example
Finding a right-inverse is essentially finding a solution \( x \) to \( f (x) = y \) systematically. Indeed, if \( g \) is a right-inverse of \( f \), then picking \( x = g (y) \) is precisely solving \( f (x) = y \) for \( x \), since \( f (x) = f (g (y)) = y \).

For example: \( f : \mathbb{R} \rightarrow \{ y \in \mathbb{R} \mid y \geq 4 \} = S \) defined by, \( \forall x \in \mathbb{R}, \ f(x) = x^2 + 4 \), has right-inverses \( g(y) = \sqrt[4]{y-4} \) and \( h(y) = -\sqrt[4]{y-4} \), where \( g : S \rightarrow \mathbb{R} \).
Indeed: \( f (g(y)) = (\sqrt[4]{y-4})^4 + 4 = (y-4) + 4 = y \)
\( f (h(y)) = (-\sqrt[4]{y-4})^4 + 4 = (y-4) + 4 = y \)

Remark (axiom of choice)
That for every collection of non-empty subsets of \( X \), say \( \{ S_i \mid i \in I \} \), there exists a "choice function" \( c : I \rightarrow X \) such that \( \forall i \in I, \ c(i) \in S_i \), is an axiom called the "axiom of choice."

Definition (left-inverse)
Let \( A, B \) be sets and let \( f : A \rightarrow B \) be a function.
If there exists a function \( g : B \rightarrow A \) such that \( g \circ f = \text{id}_A \), then we say that \( g \) is a left-inverse of \( f \), and we say that \( f \) is left-invertible.

Proposition (injectivity \( \iff \) left-invertibility)
Let \( A \) and \( B \) be sets and let \( f : A \rightarrow B \) be a function.
Then: \( f \) is injective \( \iff \) \( f \) is left-invertible.

Proof
Suppose \( f \) is left-invertible.
Then \( f \) has a left-inverse, let us call it \( g \).
Now: \( \forall x, y \in A, \ f(x) = f(y) \implies g(f(x)) = g(f(y)) \)
\[ \iff (g \circ f)(x) = (g \circ f)(y) \]
\[ \iff x = y \]
since \( g \) is a left-inverse of \( f \).
i.e. \( f \) is injective.

**Lemma 1** Suppose \( f \) is injective.

We now define \( g : B \to A \) as follows:

1. **Pick some** \( a_0 \in A \).
2. **Since** \( f \) is injective (so \( f(x) = f(y) \Rightarrow x = y \))
3. \( \forall y \in f(A) \), \( \exists ! x \in A \) s.t. \( f(x) = y \), so let \( g(y) = x \),
4. \( \forall y \in B \setminus f(A) \), let \( g(y) = a_0 \).

Then, \( \forall x \in A \), \( f(x) \in f(A) \) and hence \( g(f(x)) = x \) by definition of \( g \),

i.e. \( g \) is a left-inverse of \( f \).

\[
\text{Im} f(A)
\]

**Remark**

This new characterization of injectivity yields a new proof of "\( g \circ f \) injective \( \Rightarrow f \) injective".

Indeed: suppose \( g \circ f \) is injective, and hence left-invertible.

- Let us call its left-inverse \( h \).
- Then \( h \circ g \circ f \) is a left-inverse of \( f \), since \( (h \circ g) \circ f = h \circ (g \circ f) = \text{id} \).
- i.e. \( f \) is left-invertible, hence injective.

**Remark**

Finding a left-inverse is essentially finding a way to "systematically undo" a function, since for a left-inverse \( g \) of \( f \), \( g(f(x)) = x \).

**Definition (Inverse)**

Let \( A \) and \( B \) be sets and let \( f : A \to B \) be a function.

If there exists a function \( g : B \to A \) which is both a right-inverse and a left-inverse of \( f \), i.e. \( f \circ g = \text{id}_B \) and \( g \circ f = \text{id}_A \), then we say that \( g \) is an inverse of \( f \), we say that \( f \) is invertible, and we write \( g = f^{-1} \).
Example
Consider $f: \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{1\}$ given by $f(x) = \frac{x}{1+x}$. Then $f^{-1}(x) = \frac{y}{1-y}$.

Remark
If $f: A \to B$ is invertible, then $\forall x \in A, \forall y \in B$, $f(x) = y \iff x = f^{-1}(y)$.

Proof: Let $f: A \to B$ be invertible, let $x \in A$ and $y \in B$.
If $f(x) = y$ then $x = f^{-1}(f(x)) = f^{-1}(y)$.
If $x = f^{-1}(y)$ then $f(x) = f(f^{-1}(y)) = y$.

For example: $\frac{x}{1+x} = y \iff x = \frac{y}{1-y}$.

Theorem (invertible $\iff$ bijective)
Let $A, B$ be sets and let $f: A \to B$ be a function.
Then: $f$ is invertible $\iff$ $f$ is bijective.

Proof: $f$ bijective $\iff$ $f$ injective and surjective.
$\iff$ $f$ left-invertible and right-invertible
$\iff$ $f$ invertible

Theorem (inverse of a composition)
Let $A, B,$ and $C$ be sets and let $f: A \to B$ and $g: B \to C$ be functions.
If $f$ and $g$ are invertible, then $g \circ f$ is invertible.
Moreover, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

```
g \circ f
A ------- B ------- C
\downarrow f \downarrow \downarrow g
\downarrow \downarrow \downarrow
f^{-1} \quad \quad \quad g^{-1}
```

$f^{-1} \circ g^{-1} = (g \circ f)^{-1}$
Proof
Suppose \( f \) & \( g \) are invertible.

We want to show that

\[
\begin{align*}
(g \circ f) \circ (g^{-1} \circ f^{-1}) &= \text{id}_B \\
(f \circ g^{-1}) \circ (g \circ f) &= \text{id}_A
\end{align*}
\]

Observe

\[
(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \text{id}_B \circ g^{-1} = g \circ g^{-1} = \text{id}_c
\]

and

\[
(f \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \text{id}_B \circ f = f^{-1} \circ f = \text{id}_A
\]