Compositions and inverses

Question 1
Let \( f, g : \mathbb{Z} \to \mathbb{Z} \) be surjections. Must \( fg \) be a surjection? Prove it or find a counter-example.

Note that \( fg \) is defined by, for every \( z \in \mathbb{Z} \), \((fg)(z) = f(z)g(z)\).

Question 2
For any set \( S \), any function \( F : S \to S \), and any natural number \( n \), we define \( F^0 = \text{id}_S \) and \( F^{n+1} = F \circ F^n \).

Intuitively, \( F^n \) corresponds to composing \( F \) with itself \( n \)-times, i.e.

\[ F^n = F \circ \ldots \circ F \] (n-times)

Note that, by the principle of mathematical induction, \( F^n : S \to S \) is a well-defined function for every \( n \in \mathbb{N} \).

(i) Let \( A \) and \( B \) be sets, let \( f : A \to B \) be an invertible function, and let \( g : B \to B \) be a function. Prove that \((f^{-1} \circ g \circ f)^n = f^{-1} \circ g^n \circ f \) for every \( n \in \mathbb{N} \).

(ii) Let \( a, b \in \mathbb{R} \) and let \( h : \mathbb{R} \to \mathbb{R} \) be given by \( h(x) = a(x - b) + b \) for every \( x \in \mathbb{R} \). Use part (i) to find a formula for \( h^n \).

Question 3
Recall that a function \( f : \mathbb{R} \to \mathbb{R} \) is called strictly increasing if for every \( x, y \in \mathbb{R} \), \( x < y \Rightarrow f(x) < f(y) \).

Prove that if a strictly increasing function is invertible then its inverse is strictly increasing.

Cardinality

Question 4
Construct a countably infinite set whose elements are infinite sets of pairwise different cardinalities.

Question 5
Let \( A \) and \( B \) be sets. Suppose that \( A \cong B \). Prove that \( \mathcal{P}(A) \cong \mathcal{P}(B) \).

Question 6
Let \( A, B, C, \) and \( D \) be sets. Suppose that \( A \cong B \) and \( C \cong D \). Prove that \( A \times C \cong B \times D \).

Additional problems

Question 1 [with solutions]
Let \( A, B, C, \) and \( D \) be sets. Suppose that \( A \cong B \) and \( C \cong D \). Prove that \( A^C \cong B^D \).

Question 2 [with solutions]
Let \( A, B, \) and \( C \) be sets. Suppose that \( |A| \leq |B|, |B| \leq |C|, \) and \( |A| = |C| \). Prove that \( |A| = |B| \) and \( |B| = |C| \).
**Question 1**

Since $A \cong B$ there exists $f : A \rightarrow B$ a bijection, and since $C \cong D$ there exists $g : C \rightarrow D$ a bijection.

We want to find a bijection between $A^c$ and $B^0$.

Define $J : A^c \rightarrow B^0$ by associating to every $h \in A^c$, i.e., every function $h : C \rightarrow A$, a function $J(h) : D \rightarrow B$ defined as $J(h) = f \circ h \circ g^{-1}$.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow & & \uparrow \\
C & \xrightarrow{g^{-1}} & D \\
\end{array}
\]

Similarly, define $K : B^0 \rightarrow A^c$ by associating to every $i \in B^0$, i.e., every function $i : D \rightarrow B$, a function $K(i) : C \rightarrow A$ defined as $K(i) = f^{-1} \circ i \circ g$.

\[
\begin{array}{ccc}
A & \xleftarrow{f^{-1}} & B \\
\uparrow & & \uparrow i \\
C & \xrightarrow{g} & D \\
\end{array}
\]

We will show that $J \circ K = \text{id}_{B^0}$ and that $K \circ J = \text{id}_{A^c}$, thus showing that $J$ is invertible and hence a bijection.
First we prove that $J \circ K = \text{id}_{B^D}$.

Let $i \in B^D$, i.e. let $i : D \to B$ be a function.

Then $(J \circ K)(i) = J(f \circ i \circ g) = f \circ (f \circ i \circ g) \circ g^{-1}

= (f \circ f^{-1}) \circ i \circ g \circ g^{-1} = i

= \text{id}_{B^D}(i)

i.e. indeed $J \circ K = \text{id}_{B^D}$.

Now let us show that $K \circ J = \text{id}_{A^C}$.

Let $h \in A^C$, i.e. let $h : C \to A$ be a function.

Then $(K \circ J)(h) = K(f \circ h \circ g^{-1}) = f \circ (f \circ h \circ g^{-1}) \circ g

= (f \circ f^{-1}) \circ h \circ g \circ g^{-1} = h

= \text{id}_{A^C}(h)

i.e. indeed $K \circ J = \text{id}_{A^C}$.

Question 2

Suppose $|A| \leq |B|$, $|B| \leq |C|$, and $|A| = |C|$. We want to show that $|A| = |B|$ and $|B| = |C|$.

By Cantor-Schröder-Bernstein it is enough to show that $|B| \leq |A|$ and $|A| \leq |B|$.

Observe that $|B| \leq |C|$ and $|C| \leq |A|$, only transitivity of cardinal comparability $|B| \leq |A|, |C| \leq |A|$.

Then we know that $|A| \leq |B|$ & $|B| \leq |A|$, hence $|A| = |B|$.
and similarly we know that $|B| \leq |C|$ and $|C| \leq |B|$, hence $|B| = |C|$.