Question 1

(i) First we show that for every $c \in A/S$, $c \in A$.

This is immediate since any equivalence class of $S$ is by definition a subset of $A$.

(ii) Now we show that sets in $A/S$ are pairwise disjoint.

We want to show that $\forall c, d \in A/S$, $c \neq d \Rightarrow c \cap d = \emptyset$.

Writing the contrapositive of the implication, we then want to show that $\forall c, d \in A/S$, $c \cap d \neq \emptyset \Rightarrow c = d$.

This is precisely the implication which was proven in part (ii) of Question 3 on HW 5.1.

(iii) Finally we show that $A/S$ is a cover of $A$.

Let $a \in A$ and define $C = \{ x \in A | a S x \}$.

By reflexivity of $S$, we know that $a Sa$, i.e. $a \in C$.

By part (ii) of Question 3 on HW 5.1, we know that $C$ is an equivalence class of $S$, i.e. $C \in A/S$.

So indeed: for every $c \in A$, $\exists c \in A/S$ such that $c \in C$, which means that $A/S$ is a cover of $A$.

(ii) Let us prove that $T$ is an equivalence relation.

Reflexivity: Since $B$ is a cover of $A$, we know that for every $x \in A$,

$\exists c \in B$ such that $x \in c$, i.e. $x T x$.

Symmetry: This follows from the symmetry of "and".

Indeed, let $x, y \in A$ such that $x T y$.

Then: $x \in c$ and $y \in d \Rightarrow y \in c$ and $x \in d$, and so

in particular $y T x$.

Transitivity: Let $x, y, z \in A$ such that $x T y$ and $y T z$.

By definition of $T$, this means that there exist $d, e \in B$ such that $x \in d$, $y \in d$, $y \in e$, and $z \in e$.

In particular: $D \cap E \neq \emptyset$ (since $y \in D \cap E$), and hence

$D = E$ (since $B$ is a partition whose elements are pairwise disjoint sets).
This means that $x \in D$ and $z \in D$ for some $D \in R$, i.e. $xTz$.  

(2) Let us now prove that elements of $R$ are equivalence classes of $T$. Let $D \in R$. Since $\emptyset \notin R$, we know that $D \neq \emptyset$, i.e. $D$ is non-empty. Now pick any $x \in D$ and $y \in A$. We want to show that $xTy \Rightarrow y \in D$.

First we show that $xTy \Rightarrow y \in D$.

Suppose $xTy$, i.e. suppose $E \in R$ such that $x \in E$ and $y \in E$.

Since $x \in D \in E$, we know that $D \cap E \neq \emptyset$ since $D, E \in R$ and elements of $R$ are pairwise disjoint, it follows that $D = E$. In particular $y \in D$.

Now we show that $y \in D \Rightarrow xTy$.

Suppose $y \in D$. Since $x \in D$ and $D \in R$, it follows by definition of $T$ that $xTy$.

So indeed elements of $R$ are equivalence classes of $T$.

(3) Finally we prove that $R = A/T$.

Observe that in part (2) above we showed that $R \subseteq A/T$, so we only have to show that $A/T \subseteq R$.

Let $C \in A/T$. Since $C$ is an equivalence class, we know that it is non-empty, i.e. there exists some element $x \in C$.

Since $C$ is a coset of $A$, there exists $D \in R$ such that $x \in D$.

In particular, we proved in part (2) that $D$ is an equivalence class. So $C$ and $D$ are equivalence classes with a non-empty intersection (since $x \in C$ and $x \in D$), readily part (ii) of question 3 on HW5.1.

We know that $C = D$. Since $D \in R$, we thus know that $C \in R$.

So indeed $A/T \subseteq R$, and we can conclude that $A/T = R$. 
Question 2

Let \( B \subseteq A \) such that \( B \) is non-empty.
We want to show that \( B \) has a unique minimal element.

Existence: Since \( R \) is a well-ordered and \( B \) is non-empty,
we know that \( B \) has a minimal element.

Uniqueness: Suppose \( y, z \in B \) are minimal elements of \( B \).
Since \( y \) is a minimal element of \( B \) and since \( z \in B \),
it follows that \( y R z \).
Similarly, since \( z \) is a minimal element of \( B \) and since \( y \in B \),
it follows that \( z R y \).
So finally, by anti-symmetry of \( R \), we can conclude that \( y = z \).

Question 3

(a) \( f : \mathbb{Z} \rightarrow \mathbb{Z} \), \( \forall x \in \mathbb{Z}, f(x) = \sqrt{x + 1} \).

(ii) \( f : \mathbb{N} \rightarrow \mathbb{N} \), \( \forall x, y \in \mathbb{N}, f(x, y) = \frac{1}{2} (x + y) \).

(iii) \( f : \mathcal{P}(x) \rightarrow \mathcal{P}(x) \), \( \forall S \subseteq X, f(S) = X \setminus S = S^c \).

Question 4

(i) \( f \) is not well-defined since \( 0 \in A \) but \( f(0) = -3 \notin A \).

(ii) \( g \) is ill-defined since \( (2, 1) \in \mathbb{Z} \times \mathbb{Z} \) but \( g(2, 1) = \frac{1}{2} \cdot 3 \cdot 1 = \frac{3}{2} \notin \mathbb{N} \).

(iii) Observe that:
\[
\begin{align*}
l_1(1) &= (-1)^7 = -1 = 1 \times 1(1)
\end{align*}
\]
\[
\begin{align*}
l_2(0) &= \frac{0}{2} = 0 = 1 \times 2(0)
\end{align*}
\]
\[
\begin{align*}
l_3(1) &= 1^3 = 1 = 1 \times 3(1)
\end{align*}
\]

i.e., \( \forall x \in B, l(x) = 1 \times x(1), \) i.e. indeed \( l = x \times 1(B) \).

(iv) Let \((x, y) \in A \). Then:
\[
\begin{align*}
ed(x, y) &= \frac{x^3 - y^3}{x - y} \\
&= \frac{x(x^2 + xy + y^2) - y(x^2 + xy + y^2)}{x - y} \\
&= \frac{(x - y)(x^2 + xy + y^2)}{x - y} \\
&= x^2 + xy + y^2
\end{align*}
\]
\[
(x+y)^2 - xy = \begin{cases} 
0 & \text{since } (x, y) \in A \iff x - y \neq 0 \\
}\end{cases}
\]

\[
= j(x, y)
\]

i.e. indeed \( i = j \).

**Question 5**

(i) \( \operatorname{Im} f(S) \) is the set of non-negative multiples of 3, i.e. \( \operatorname{Im} f(S) = \{ 3m \mid m \in \mathbb{N} \} = \{ x \in \mathbb{N} \mid x \text{ is a multiple of } 3 \} = \{ 0, 3, 6, 9, \ldots \} \).

(ii) \( \operatorname{Im} f(S) \) is the diagonal of \( X \), i.e. \( \operatorname{Im} f(S) = \{ (x, x) \in X \times X \mid x \in X \} \).

(iii) \( \operatorname{Im} f(S) = \{ 1, 3 \} \).

(iv) \( \operatorname{PreIm} f(S) = \{ 1, 2, 3 \} \).

(v) \( \operatorname{PreIm} f(S) = \{ 1, 2, 3 \} \).

(vi) \( \operatorname{PreIm} f(S) = \mathbb{N} \).

**Question 6**

We show directly that \( \operatorname{Im} f(S \cup T) = \operatorname{Im} f(S) \cup \operatorname{Im} f(T) \), i.e. we show that both (i) & (ii) hold.

Let \( y \in B \). Then:

\[
\begin{align*}
\text{let } y & \in \operatorname{Im} f(S \cup T) \\
\iff & \exists x \in S \cup T \text{ s.t. } f(x) = y \\
\iff & \left( \exists x \in S \text{ s.t. } f(x) = y \right) \lor \left( \exists x \in T \text{ s.t. } f(x) = y \right) \\
\iff & y \in \operatorname{Im} f(S) \lor y \in \operatorname{Im} f(T) \\
\iff & y \in \operatorname{Im} f(S) \cup \operatorname{Im} f(T) \\
\end{align*}
\]

i.e. indeed \( \operatorname{Im} f(S \cup T) = \operatorname{Im} f(S) \cup \operatorname{Im} f(T) \).
Question 7

(i) The claim is false.

Consider $A = \{1, 2\}$, $B = \{a, b\}$, $f(1) = f(2) = e$, and $S = \{1\}$.

Then: $\text{Inf}(S^c) = \text{Inf}(\{1\}) = \{e\}$

$$\left(\text{Inf}(S)^c\right)^c = \{e\}^c = \emptyset$$

and indeed $\text{Inf}(S^c) = \{e\} \neq \emptyset = \left(\text{Inf}(S)^c\right)^c$.

(ii) The claim is false.

Consider $A = \{1, 3\}$, $B = \{a, b, 3\}$, $f(1) = a$, and $S = \{3\} \cup \{1\}$.

Then: $\left(\text{Inf}(S)^c\right)^c = \{a\}^c = \{b\}$

$$\text{Inf}(S^c) = \text{Inf}(\emptyset) = \emptyset$$

and indeed $\left(\text{Inf}(S)^c\right)^c = \{b\} \neq \emptyset = \text{Inf}(S^c)$.

Question 8

The claim is false.

Consider $A = \{1, 3\}$, $B = \{a, b, 3\}$, $f(1) = a$, and $Y = \{3\}$.

Then: $B \cap \text{Inf}(Y) = \emptyset$, hence $\text{Inf}(B \cap \text{Inf}(Y)) = \text{Inf}(\emptyset) = \emptyset$,

i.e. $Y = 3 \not\in \emptyset \neq \text{Inf}(B \cap \text{Inf}(Y))$. 