Equivalence relations

Question 1 [5 points]
The goal of this problem is to show that partitions and quotients are essentially the same thing. More precisely, you will prove that (i) quotients are partitions, and (ii) given a partition, there exists an equivalence relation whose corresponding quotient is equal to the partition given.

(i) Let $A$ be a set and let $S$ be an equivalence relation on $A$. Prove that $A/S$ is a partition of $A$.

(ii) Let $R$ be a partition of $A$, and suppose that $\emptyset \not\in R$. Define a relation $T$ on $A$ via:

$$\forall x, y \in A, \quad xTy \iff \exists D \in R \text{ such that } x \in D \text{ and } y \in D$$

Prove that $T$ is an equivalence relation, prove that every element of $R$ is an equivalence class of $T$, and prove that $R = A/T$.

Remember that you are allowed to use any results proven in earlier homework problem sets.

Order relations

Question 2 [5 points]

A partial order $R$ on a set $A$ is called a well-order if every non-empty subset of $A$ has a minimal element with respect to $R$. Note that the well-ordering principle states that $\leq$ is a well-order on $\mathbb{N}$.

Let $A$ be a set and let $R$ be a well-order on $A$. Prove that every non-empty subset of $A$ has a unique minimal element with respect to $R$. In other words, prove that

$$\forall B \subseteq A, \quad B \neq \emptyset \implies \exists! x \in B \text{ such that } \forall y \in B, xRy$$

Functions

Question 3 [5 points]

Below you are asked to “use proper notation” to define a few functions. For example, if you are asked to define a function that inputs a natural number and outputs its square, using proper notation, you could write:

$$f : \mathbb{N} \to \mathbb{N} \text{ defined via: } \forall n \in \mathbb{N}, \ f(n) = n^2$$

or

$$f : \mathbb{N} \to \mathbb{R} \text{ defined via: } \forall x \in \mathbb{N}, \ f(x) = x^2$$

(i) Use proper notation to define a function that inputs an integer and outputs the square root of its absolute value.

(ii) Use proper notation to define a function that inputs a pair of natural numbers and outputs their average (arithmetic mean).

(iii) Let $X$ be a set. Use proper notation to define a function that inputs a subset of $X$ and outputs that set’s complement (where the universal set is taken to be $X$).

Question 4 [5 points]

(i) Let $A = \{-2, -1, 0, 1, 2\}$. Let $f : A \to A$ be defined by $\forall x \in A, \ f(x) = x^2 - 3$. Is $f$ well-defined? Explain why or why not.

(ii) Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be defined by $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}, \ g(x, y) = \frac{1}{2}|x + 1| \cdot |y|$. Is $g$ well-defined? Explain why or why not.
(iii) Let \( B = \{-1, 0, 1\} \), and let \( h : B \to B \) be defined by \( \forall x \in B, \ h(b) = b^7 \). Show that \( h = \text{id}_B \).

(iv) Let \( A = \{(x, y) \in \mathbb{R}^2 \mid x \neq y\} \), and let \( i, j : A \to \mathbb{R} \) be defined by \( \forall (x, y) \in A, \ i(x, y) = \frac{x^3 - y^3}{x+y} \) and \( j(x, y) = (x+y)^2 - xy \). Show that \( i = j \).

### Images and pre-images

**Question 5 [5 points]**

For each of the following functions \( f \) and subsets \( S \) of their domain, describe \( \text{Im}_f (S) \).

(i) \( f : \mathbb{Z} \to \mathbb{Z} \) defined by \( f(n) = 3n \), with \( S = \mathbb{N} \).

(ii) \( f : X \to X \times X \) (where \( X \) is any set) defined by \( f(x) = (x, x) \), with \( S = X \).

(iii) \( f : \{a, b, c\} \to \{1, 2, 3\} \) defined by \( f(a) = 1, \ f(b) = 3, \) and \( f(c) = 1 \), with \( S = \{a, b, c\} \).

Now let \( f : \mathbb{Z} \to \mathbb{Z} \) be a function defined by \( f(x) = x^2 \) for all \( x \in \mathbb{Z} \). For each of the following set \( S \), describe \( \text{PreIm}_f (S) \).

(iv) \( S = \{1, 4, 9\} \).

(v) \( S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \).

(vi) \( S = \mathbb{N} \).

**Question 6 [5 points]**

Let \( f : A \to B \) be a function. Let \( S, T \subseteq A \). For each of the following claims, prove that it must hold, or disprove it by finding a counterexample.

(i) \( \text{Im}_f (S \cup T) \subseteq \text{Im}_f (S) \cup \text{Im}_f (T) \)

(ii) \( \text{Im}_f (S) \cup \text{Im}_f (T) \subseteq \text{Im}_f (S \cup T) \)

**Question 7 [5 points]**

Let \( f : A \to B \) be a function. Let \( S, T \subseteq A \). For each of the following claims, prove that it must hold, or disprove it by finding a counterexample.

(i) \( \text{Im}_f (S^c) \subseteq (\text{Im}_f (S))^c \)

(ii) \( (\text{Im}_f (S))^c \subseteq \text{Im}_f (S^c) \)

Here complements are taken with respect to \( A \) for subsets of \( A \) and with respect to \( B \) for subsets of \( B \).

**Question 8 [5 points]**

Prove or disprove the following claim:

Let \( A \) and \( B \) be sets. Let \( f : A \to B \) be a function. Let \( Y \subseteq B \). Then \( Y \subseteq \text{Im}_f (\text{PreIm}_f (Y)) \)
Additional problems

Question 1 [with solutions]
Use proper notation to define a function that inputs an even natural number and outputs half of that number.

Question 2 [with solutions]
(i) Let \( f = \{ (a^2, a) \mid a \in \mathbb{R} \} \). Is \( f \) a well-defined function? Explain why or why not.
(ii) Let \( f = \{ (x, y) \in \mathbb{Q} \times \mathbb{Q} \mid x, y \in \mathbb{Q}, xy = 1 \} \). Is \( f \) a well-defined function? Explain why or why not.

Question 3 [with solutions]
Let \( f : \mathbb{N} \to \mathbb{R} \) be a function defined by \( f(x) = \sqrt{x} \) for all \( x \in \mathbb{N} \). Describe

(i) \( \text{Im}_f(\{0, 1, 2, 3, 4\}) \)
(ii) \( \text{PreIm}_f(\mathbb{Z}) \)
(iii) \( \text{PreIm}_f(\mathbb{R}) \)

Question 4 [with solutions]
Let \( f : A \to B \) be a function. Let \( S, T \subseteq B \). For each of the following claims, prove that it must hold, or disprove it by finding a counterexample.

(i) \( \text{PreIm}_f(S \cup T) \subseteq \text{PreIm}_f(S) \cup \text{PreIm}_f(T) \)
(ii) \( \text{PreIm}_f(S) \cup \text{PreIm}_f(T) \subseteq \text{PreIm}_f(S \cup T) \)

Question 5 [with solutions]
Prove or disprove the following claim:
Let \( A \) and \( B \) be sets. Let \( f : A \to B \) be a function. Let \( X \subseteq A \). Then \( X \subseteq \text{PreIm}_f(\text{Im}_f(X)) \)
Question 1
Let $E = \{\text{even natural numbers}\} = \{m \in \mathbb{N} \mid \exists a \in \mathbb{N}, \text{ s.t. } m = 2a\}$
and define $f : E \to \mathbb{N}$ by $\forall e \in E, f(e) = \frac{e}{2}$.

Question 2
(i) $f$ is not a well-defined function because it is not left-total.
For example, there is no $a \in \mathbb{R}$ such that $(-1)f(a)$, since $-1 \neq a^2$ \forall \mathbb{R}.

(ii) $f$ is not a well-defined function because it is not left-total.
In particular, there is no $y \in \mathbb{R}$ such that $0f(y)$, since $0 \neq 1 \forall y \in \mathbb{R}$.

Question 3
(i) $\text{Im } f(\{0, 1, 2, 3, 4\}) = \{0, 1, \sqrt{2}, \sqrt{3}, 2\}$.
(ii) $\text{PreIm } f(\mathbb{N}) = \{\text{perfect squares}\} = \{n^2 \mid n \in \mathbb{N}\} = \{0, 1, 4, 9, \ldots\}$
(iii) $\text{PreIm } f(\mathbb{R}) = \mathbb{N}$

Question 4
Both (i) and (ii) are true.
We prove that $\text{PreIm } f(S \cup T) = \text{PreIm } f(S) \cup \text{PreIm } f(T)$
by proving that $\forall x \in A$, $x \in \text{PreIm } f(S \cup T)$
\iff $x \in \text{PreIm } f(S) \cup \text{PreIm } f(T)$.

Let $x \in A$. Then:
\[ \iff f(x) \in S \cup T \]
\[ \iff f(x) \in S \text{ or } f(x) \in T \]
\[ \iff x \in \text{PreIm } f(S) \text{ or } x \in \text{PreIm } f(T) \]

Question 5
The claim is true.
Let $e \in \mathbb{R}$. Then $\frac{e}{f(e)} \in \text{Im } f(X)$ by definition of images,
and hence $e \in \text{PreIm } f(\text{Im } f(X))$ since $f(x) = 0$
\[ \iff y \in \text{PreIm } f(\text{Im } f(X)) \text{ for every } y \in A \text{ and } f(x) = 0 \text{ for every } x \in \mathbb{R} \]
\[ (\text{here } D = \text{Im } f(X)). \]