A new proof of the 2-dimensional Halpern–Läuchli Theorem

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January 2017

Abstract

We provide an ultrafilter proof of the 2-dimensional Halpern–Läuchli Theorem in the following sense. If $T_0$ and $T_1$ are trees and $T_0 \otimes T_1$ denotes their level product, we exhibit an ultrafilter $\mathcal{U} \in \beta(T_0 \otimes T_1)$ so that every $A \in \mathcal{U}$ contains a subset of the form $S_0 \otimes S_1$ for suitable strong subtrees of $T_0$ and $T_1$. We then discuss obstacles to extending our method of proof to higher dimensions.

1 Introduction

Our conventions on trees mostly follow [2]. By a tree $(T, \leq)$, we mean a rooted, finitely-branching tree of height $\omega$ so that each $t \in T$ has at least 2 immediate successors. If $t \in T$, we set $\text{Pred}(t, T) := \{s \in T : s \leq t\}$. The level of $t \in T$, denoted $\text{Lev}(t, T)$, is the number $|\text{Pred}(t, T)|$. If $n < \omega$, we set $T(n) := \{t \in T : \text{Lev}(t, T) = n\}$. Given $s, t \in T$, we say that $t$ is an immediate successor of $s$ in $T$ if $s \leq t$ and $\text{Lev}(t, T) = \text{Lev}(s, T) + 1$. Write $\text{IS}(s, T)$ for the immediate successors of $s$ in $T$. Note that for every $s \in T$, $2 \leq |\text{IS}(s, T)| < \omega$.

A subset $S \subseteq T$ is called a strong subtree of $T$ if $(S, \leq_{|S})$ is a tree satisfying the following two items.

1. For some increasing function $f : \omega \to \omega$, we have $S(n) \subseteq T(f(n))$.
2. For every $s \in S$ and $t \in \text{IS}(s, T)$, there is a unique $t' \in \text{IS}(s, S)$ with $t \leq t'$.

If in item (1) we have a specific $f : \omega \to \omega$ in mind, we call $S \subseteq T$ an $f$-strong subtree of $T$.

If $d < \omega$ and $T_0, ..., T_{d-1}$ are trees, the level product, denoted $T_0 \otimes \cdots \otimes T_{d-1}$, is the set $\bigcup_n T_0(n) \times \cdots \times T_{d-1}(n)$, which receives a tree structure in the obvious way.

We are now ready to state the Halpern–Läuchli Theorem [1].

2010 Mathematics Subject Classification. Primary: 05D10; Secondary: 03E02, 54D80.
Key words and phrases. Halpern–Läuchli Theorem, ultrafilters
The author was partially supported by NSF Grant no. DGE 1252522.
Theorem 1.1 (Halpern–Läuchli). Let $d < \omega$, and let $T_0, \ldots, T_{d-1}$ be trees. Let $\chi : T_0 \otimes \cdots \otimes T_{d-1} \to 2$ be a coloring. Then there are an increasing function $f : \omega \to \omega$ and $f$-strong subtrees $S_i \subseteq T_i$ so that $S_0 \otimes \cdots \otimes S_{d-1}$ is monochromatic for $\chi$.

The parameter $d$ is referred to as the dimension. We will provide a new proof of the Halpern–Läuchli Theorem for $d = 2$.

2 Warmup in one dimension

As a warmup, we will first provide a new proof for $d = 1$. If $T$ is a tree, a branch through $T$ is a maximal linearly ordered subset of $T$. If $x \subseteq T$ is a branch, then for every $n < \omega$, there is a unique element of $T(n)$ in $x$, which we will denote by $x(n)$. Let $[T]$ denote the set of branches through $T$. We endow $[T]$ with the topology generated by the sets $(N_t : t \in T)$, where for $t \in T$, we set $N_t := \{x \in [T] : t \in x\}$. With this topology, $[T]$ is homeomorphic to Cantor space. Of particular interest will be the ideal of nowhere dense subsets of $[T]$.

For $X$ any nonempty set, let $\beta X$ denote the set of ultrafilters on $X$. In particular, let $\beta([T])$ denote the set of ultrafilters on $[T]$, where we now view $[T]$ as just a set. Fix $U \in \beta([T])$ avoiding the nowhere dense ideal. Also fix any nonprincipal ultrafilter $V \in \beta \omega$.

We define the ultrafilter $U \otimes V \in \beta T$ as follows. If $A \subseteq T$, we have

$$A \in U \otimes V \Leftrightarrow \forall^U x \in [T] \forall^V n < \omega \ (x(n) \in A).$$

Fix $A \in U \otimes V$. We will show that $A$ contains a strong subtree of $T$. To see this, first set

$$A_V := \{x \in [T] : \{n < \omega : x(n) \in A\} \in V\}$$

By the definition of $U \otimes V$, we have $A_V \in U$. As $U$ avoids the nowhere dense ideal, $A_V$ is somewhere dense. This means that for some $t \in T$, $A_V$ is dense in $N_t$. Pick any $x \in A_V$ with $t \in x$. Then $\{n < \omega : x(n) \in A\} \in V$. So for some $n < \omega$, we have $t \leq x(n)$ and $x(n) \in A$. Set $S(0) = \{x(n)\}$.

Assume $S(m) = \{s_0, \ldots, s_{k-1}\}$ has been determined. Let $\bigcup_{i < k} IS(s_i, T) = \{t_0, \ldots, t_{\ell-1}\}$. For each $i < \ell$, we can find $x_i \in A_V$ with $t_i \in x_i$. Then $\bigcap_{i < \ell} \{n < \omega : x_i(n) \in A\} \in V$. So for some suitably large $n$, set $S(m + 1) = \{x_i(n) : i < \ell\}$.

3 The proof for 2 dimensions

The proof for $d = 2$ will be very similar to the proof for $d = 1$. We will choose ultrafilters $U \in \beta([T_0 \otimes T_1])$ and $V \in \beta \omega$ and form $U \otimes V$ as before, and argue that every $A \in U \otimes V$ contains a subset of the form $S_0 \otimes S_1$ for some $f$-strong subtrees $S_0$ and $S_1$. The added difficulty in dimension 2 is that we must choose $U$ more carefully.

Notice first that $[T_0 \otimes T_1] \cong [T_0] \times [T_1]$. Let $\pi_i : [T_0] \times [T_1] \to [T_i]$ be the projection maps. We call $Z \subseteq [T_0] \times [T_1]$ a dense-by-dense-filter, or DDF for short, if
1. \( \pi_0(Z) \subseteq [T_0] \) is dense.

2. Letting \( (Z)_x = \{y \in [T_1] : (x, y) \in z\} \), the collection \( \{(Z)_x : x \in \pi_0(Z)\} \) generates a filter of dense subsets of \([T_1]\).

If \( s \in T_0 \) and \( t \in T_1 \), we say \( Z \subseteq N_s \times N_t \) is \((s, t)\)-DDF if the relativized analogs of items (1) and (2) hold. We call \( Z \subseteq [T_0] \times [T_1] \) somewhere DDF if \( Z \) is \((s, t)\)-DDF for some \( s \in T_0 \) and \( t \in T_1 \).

**Proposition 3.1.** The collection of somewhere DDF subsets of \([T_0] \times [T_1]\) is weakly partition regular, i.e. for any \( k < \omega \) and partition \([T_0] \times [T_1] = P_0 \cup \cdots \cup P_{k-1} \), some \( P_k \) contains a somewhere DDF subset.

**Proof.** We prove a “relativized” version. First suppose that \( X \subseteq [T_0] \) is non-meager and \( Y \subseteq [T_1] \) is somewhere dense. By zooming in to a suitable \( N_s \subseteq [T_0] \) and \( N_t \subseteq [T_1] \), we may assume that \( X \subseteq [T_0] \) is nowhere meager and \( Y \subseteq [T_1] \) is dense.

Fix a partition \( X \times Y = P_0 \cup \cdots \cup P_{k-1} \). We will attempt to find \( D \subseteq P_0 \) which is DDF. Enumerate \( T_0 := \{s_n : n < \omega\} \). First set \( Y = Y_0 \). At stage \( k \), starting with \( k = 0 \), we find if possible some \( x_k \in X \cap N_{s_k} \) so that \( Y_{k+1} := (P_0)_{x_k} \cap Y_k \) is dense. If we can do this for every \( k < \omega \), then \( P_0 \) contains a DDF subset as desired.

Suppose we fail at stage \( k \). This means that for every \( x \in X \cap N_{s_k} \), there is some \( t_x \in T_1 \) so that \( (P_0)_x \cap Y_k \cap N_{t_x} = \emptyset \). Since \( X \) is nowhere meager, there is some \( t \in T_1 \) so that \( X' := \{x \in X \cap N_{s_k} : t_x = t\} \) is non-meager. Setting \( Y' = Y_k \cap N_t \), we have \( X' \) non-meager, \( Y' \) somewhere dense, and the partition relative to \( X' \times Y' \) has one fewer piece. \( \square \)

We can now complete the proof of Halpern–Läuchli for \( d = 2 \). Let \( U \in \beta([T_0] \times [T_1]) \) be an ultrafilter chosen so that every large set contains a somewhere DDF subset. Let \( V \in \beta \omega \) be any non-principal ultrafilter, and define \( U \otimes V \) exactly as before.

Fix \( A \in U \otimes V \). We will show that \( A \) contains a subset of the form \( S_0 \otimes S_1 \) for suitable strong subtrees \( S_0 \subseteq T_0 \) and \( S_1 \subseteq T_1 \). First set

\[
A_V := \{(x, y) \in [T_0] \times [T_1] : \{n < \omega : (x(n), y(n)) \in A\} \in V\}.
\]

By definition of \( U \otimes V \), we have \( A_V \in U \). Let \( D \subseteq A_V \) be an \((s, t)\)-DDF subset for some \( s \in T_0 \) and \( t \in T_1 \). Pick some \((x, y) \in D\); then \( \{n < \omega : (x(n), y(n)) \in A\} \in V \). Pick \( n > \max(\text{Lev}(s, T_0), \text{Lev}(t, T_1)) \) with \((x(n), y(n)) \in A\), and set \( S_0(0) = \{x(n)\}, S_1(0) = \{y(n)\} \).

Assume \( S_0(m) = \{s_0, ..., s_{k_0-1}\} \) and \( S_1(m) = \{t_0, ..., t_{k_1-1}\} \) have been determined. Let \( \bigcup_{i < k_0} \text{IS}(s_i, T_0) = \{s'_0, ..., s'_{k_0-1}\} \). For each \( i < \ell_0 \), we can find \( x_i \in \pi_0(D) \) with \( s'_i \subset x_i \). Since \( D \) is \((s, t)\)-DDF, the set \( \bigcap_{i < \ell_0} (D)_{x_i} \) is dense in \( N_t \). Let \( \bigcup_{i < k_1} \text{IS}(t_i, T_1) = \{t'_0, ..., t'_{k_1-1}\} \). For each \( j < \ell_1 \), we can find \( y_j \in [T_1] \) so that \((x_i, y_j) \in D\) for each \( i < \ell_0 \). Now observe that \( \{n < \omega : \forall i < \ell_0 \forall j < \ell_1 (x_i(n), y_j(n)) \in A\} \in V \). For a suitably large \( n \), set \( S_0(m + 1) = \{x_i(n) : i < \ell_0\} \) and \( S_1(m + 1) = \{y_j(n) : j < \ell_1\} \).
4 Obstacles to higher dimensions

In this last section, we show that the appropriate notion of “somewhere DDF” subset of $2^\omega \times 2^\omega \times 2^\omega$ is consistently not weakly partition regular, which prevents the proof for $d = 2$ from being generalized. To be precise, let us call the notion of DDF from the last section DDF(2) to emphasize the dimension.

For $i < j < 3$, let $\pi_{i,j} : (2^\omega)^3 \to 2^\omega \times 2^\omega$ be the corresponding projection. Let us call $Z \subseteq (2^\omega)^3$ DDF(3) if the following conditions are met.

1. $\pi_{0,1}(Z) \subseteq 2^\omega \times 2^\omega$ is DDF(2).
2. Letting $(Z)(x,y) = \{ z \in 2^\omega : (x,y,z) \in Z \}$, the collection $\{(Z)(x,y) : (x,y) \in \pi_{0,1}(Z)\}$ generates a filter of dense subsets of $2^\omega$.

The notion of somewhere DDF(3) is defined similarly to the last section.

**Proposition 4.1.** ZFC does not prove that the collection of somewhere DDF(3) subsets of $(2^\omega)^3$ is weakly partition regular. In particular, under CH there is a $2$-coloring of $(2^\omega)^3$ so that neither color class contains a somewhere DDF(3) subset.

**Proof.** For each $n \in \omega$, let $B_n = \{ z \in 2^\omega : z(n) = 0 \}$. Our coloring $(2^\omega)^3 = P_0 \cup P_1$ will be such that for every $x,y \in 2^\omega$, we have $(P_0)(x,y) = B_n$ for some $n$. So our construction is just to describe the map $\varphi : 2^\omega \times 2^\omega \to \omega$ so that $(P_0)(x,y) = B_{\varphi(x,y)}$. We will more-or-less use an Ulam matrix to describe $\varphi$.

Identify $2^\omega$ with $\omega_1 \setminus \omega$. For each infinite ordinal $\alpha < \omega_1$, let $f_\alpha : \alpha \to \omega \setminus \{0\}$ be a bijection. We then set

$$\varphi(\alpha, \beta) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ f_\alpha(\beta) & \text{if } \beta < \alpha, \\ f_\beta(\alpha) & \text{if } \alpha < \beta \end{cases}$$

For any distinct infinite $\alpha_0, \alpha_1 < \omega_1$ and $n < \omega$, there is at most 1 ordinal $\beta$ with $\varphi(\alpha_0, \beta) = \varphi(\alpha_1, \beta) = n$.

Now suppose $D \subseteq P_0$ is $(s,t,u)$-DDF(3) for some $s,t,u \in 2^{<\omega}$. This implies that for some $n$ and every $(x,y) \in \pi_{0,1}(D)$, we have $\varphi(x,y) < n$. For any $N < \omega$, we can find $\{x_i : i < N\}$ and $\{y_i : i < N\}$ so that $(x_i,y_j) \in \pi_{0,1}(D)$ for each $i, j < N$. By making $N$ large enough, we can find $x'_0, x'_1$ and $y'_0, y'_1$ so that for every $i, j < 2$, we have $\varphi(x'_i, y'_j) = k$ for some fixed $k$. This is a contradiction. $\square$

**References**


4I thank MathOverflow user Ashutosh [3] for suggesting the use of an Ulam matrix. Ashutosh also provides evidence which suggests that working with MA + $\mathfrak{c} = \omega_2$ might provide a different outcome.