Unit Test 2 Review Problems – Set A

We have chosen these problems because we think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on the exam will resemble these problems in any way whatsoever. Remember that on exams you will have to supply evidence for your conclusions and may have to explain why your answers are reasonable and appropriate.

1. For each of the equations listed below, interpret the equation as a surface in 3D. Describe the surface and draw a sketch of its graph.

(a) $3x + 2y + 10z = 20$. **(b)** $x^2 + y^2 = 9$.

(c) $xy = 4$ **(d)** $z = 4x^2 + y^2$

2. For each of the limits listed below, use the suggested strategy to show that the limit does not exist.

(a) **Limit:** $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$

Strategy: Substitute $y = mx$ and take limit as $x \to 0$.

(b) **Limit:** $\lim_{(x,y,z)\to(0,0,0)} \frac{x + y + z}{x^2 + y^2 + z^2}$

Strategy: Approach (0, 0, 0) along a coordinate axis from the positive and negative directions.

(c) **Limit:** $\lim_{(x,y)\to(0,0)} \frac{x^2 - 2y^2}{x^2 + y^2}$

Strategy: Draw a contour plot.

(d) **Limit:** $\lim_{(x,y)\to(1,1)} \frac{x - y}{x^3 - y}$

Strategy: Make a table.

3. A one-meter long metal bar is heated unevenly with temperature $H(x, t)$ (measured in $^{\circ}$C) depending on the distance x (in meters) from the left end of the bar and the time t

(measured minutes) since the heat source is removed. The temperature is given by the formula:

$$H(x,t) = 100 \cdot e^{-0.1t} \cdot \sin(\pi x) \qquad 0 \le x \le 1 \qquad \text{and} \qquad t \ge 0.$$

(a) Sketch graphs showing $H(x, t)$ versus x for the instants in time $t = 0$ and $t = 1$.

(b) Calculate the value of the partial derivative $H_x(0.2, t)$. What is the practical interpretation of this partial derivative? Briefly explain why the sign (+ or −) of the partial derivative makes sense in terms of the practical situation that $H(x, t)$ represents.

(c) Calculate the value of the partial derivative $H_x(0.8, t)$. What is the practical interpretation of this partial derivative? Briefly explain why the sign (+ or −) of the partial derivative makes sense in terms of the practical situation that $H(x, t)$ represents.

(d) Calculate $H_t(x, t)$. What is its sign (+ or −)? Give a practical interpretation of this partial derivative in terms of the situation that $H(x, t)$ represents.

4. Find an equation for the tangent plane at the point given for each of the surfaces $z = f(x, y)$ listed below.

(a) **Surface:** $z = x^2 + y^2$

Point: (3, 4, 25)

(b) **Surface:** $z = \sin\left(\frac{1}{2}\pi xy\right)$

Point: (3, 5, −1)

(c) **Surface:** $z = x^3 - y^3$

Point: (3, 2, 19)

(d) **Surface:** $z = xy$

Point: (1, −1, −1)

5. In this problem you will be concerned with the function $g(x,y) = \sqrt{x^2 + 3y + 3}$.

(a) Calculate $\nabla g(1,4)$.

(b) Find a formula for the linear approximation of $g(x, y)$ based at the point $(x, y) = (1, 4)$.

(c) Use your answer to Part (b) to approximate the value of $g(1.01, 3.98)$.

6. Suppose that the temperature T (in ^{o}C) at a point (x, y, z) in space is given by the function:

$$W = 50 + xyz.$$

In this problems all distances are considered to be in meters.

(a) Find the rate of change of temperature with respect to distance at the point (3, 4, 1) in the direction of the vector <1, 2, 2>.

(b) Find the unit vector **u** that gives the maximum value of $D_{\mathbf{u}}W(3, 4, 1)$.

(c) What is the maximum rate of change of temperature with respect to distance at the point (3, 4, 1)?

7. Find and classify all critical points for each of the surfaces listed below.

(a) $f(x, y) = 2x^2 + y^2 + 4x - 4y + 5.$

(b) $f(x, y) = 2x^2 - 3y^2 + 2x - 3y + 7.$

(c) $f(x, y) = 2x^2 + 2xy + y^2 + 4x - 2y + 1.$

(d) $f(x, y) = x^3 + y^3 + 3xy + 3.$

8. In physics and chemistry the equation of state of a liquid or gas with volume V, pressure p and temperature T is an equation of the form:

$$F(p, V, T) = 0.$$

For example, for an ideal gas (that is, a gas with no intermolecular interactions beyond elastic collisions), the equation of state is:

$$pV - nRT = 0,$$

where R is a constant and n is the number of moles of gas present. The **thermal expansivity** of the liquid, α, is defined to be:

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T}$$

and the **isothermal compressivity** of the liquid, β, is defined to be:

$$\beta = \frac{-1}{V}\frac{\partial V}{\partial p}.$$

(a) Show that $\dfrac{\partial p}{\partial T} = \dfrac{\alpha}{\beta}$.

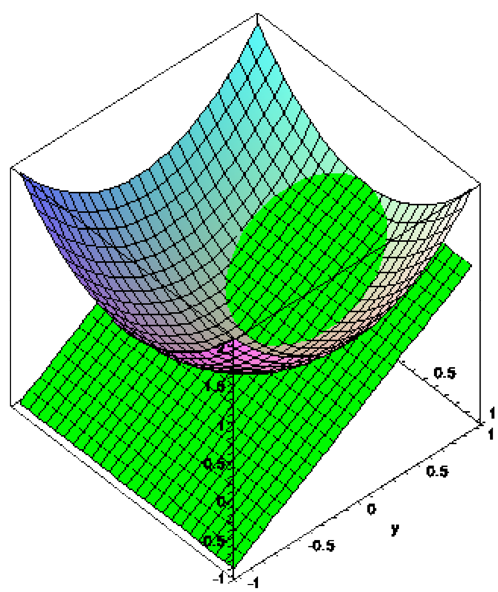
(b) The thermal expansivity and isothermal compressivity of liquid mercury are $\alpha = 1.8\times10^{-4}$ and $\beta = 3.9\times10^{-6}$ in liters/atmospheres/°C units. Suppose that a thermometer bulb is completely filled with mercury at 50°C. If the thermometer can withstand an internal pressure of no more than 200 atmospheres, can the thermometer be heated to 55°C without breaking?

9. For each of the scenarios described below, compute the rate of change of the volume with respect to time.

(a) A rectangular block of ice is melting. The base of the block of ice has a square shape. When the height of the block is 1 foot and each edge of its square base is 2 feet in length, the height is decreasing at a rate of 2 inches per hour and each edge of the square base is decreasing at a rate of 3 inches per hour.

(b) Sand falling from a conveyor belt forms a pile in the shape of a cone. When the pile has a height of 5 feet and the radius of its base is two feet, the height is increasing at 0.4 feet per minute and the radius of the base is increasing at 0.7 feet per minute.

10. Consider the curve formed by the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$. This curve is pictured below.



(a) Find a formula for the vector function that describes this curve.

(b) Show that the projection of this curve onto the *xy*-plane forms a circle.