Outline

1. Tangent lines for polar curves.

2. Areas for polar curves.

3. Arc length for polar curves.

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See Stella in 6116 Wean to sign up.

1. Tangent Lines for Polar Curves

- Assume curve described by an equation like:

$$r = f(\theta).$$

- **Want:** tangent line equation in terms of x, y.

Coordinates of tangency point

$$x = r \cdot \cos(\theta) = f(\theta) \cdot \cos(\theta)$$

$$y = r \cdot \sin(\theta) = f(\theta) \cdot \sin(\theta)$$

Slope of tangent Line

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \qquad \frac{\cancel{(f'(\theta))(\cos(\theta)) + f(\theta)\sin(\theta)}}{}$$

$$= \frac{f'(\theta)\cdot\sin(\theta) + f(\theta)\cdot\cos(\theta)}{f'(\theta)\cdot\cos(\theta) - f(\theta)\cdot\sin(\theta)}$$

Example

Find the tangent line to:

$$r = 3\cdot\sin(\theta)$$

at $\theta = \pi/6$.

Solution

Point of tangency:

$$x(\pi/6) = 3\cdot\sin(\pi/6)\cdot\cos(\pi/6)$$

$$= \frac{3\sqrt{3}}{4}$$

$$y(\pi/6) = 3\cdot\sin^2(\pi/6)$$

$$= \frac{3}{4}$$

Slope:

$$\frac{dy}{d\theta} = 6 \cdot \sin(\theta) \cdot \cos(\theta)$$

$$\left.\frac{dy}{d\theta}\right|_{\theta=\pi/6} = \frac{6\sqrt{3}}{4}$$

$$\frac{dx}{d\theta} = 3\cos^2(\theta) - 3\sin^2(\theta)$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = \frac{9}{4} - \frac{3}{4} = \frac{6}{4}$$

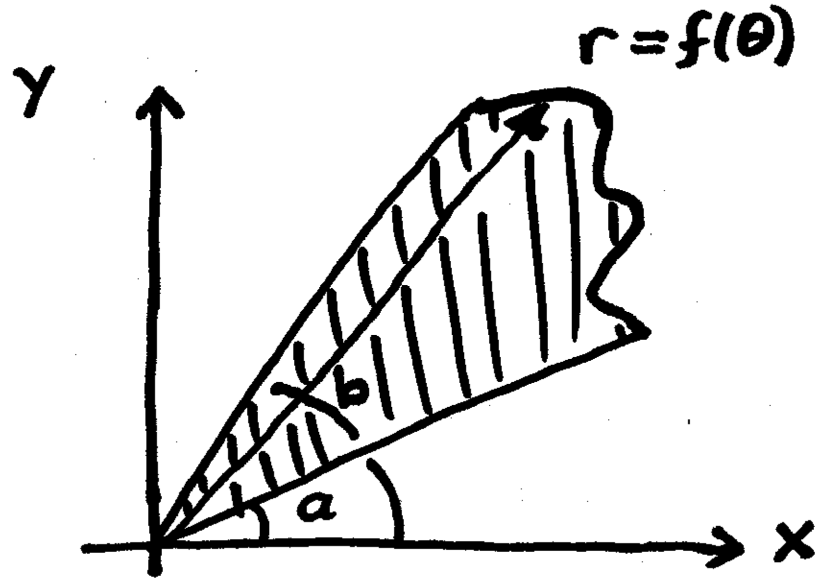
$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \sqrt{3}$$

Final answer:

$$y - \frac{3}{4} = \sqrt{3}\left(x - \frac{3\sqrt{3}}{4}\right)$$

2. Areas with Polar Curves

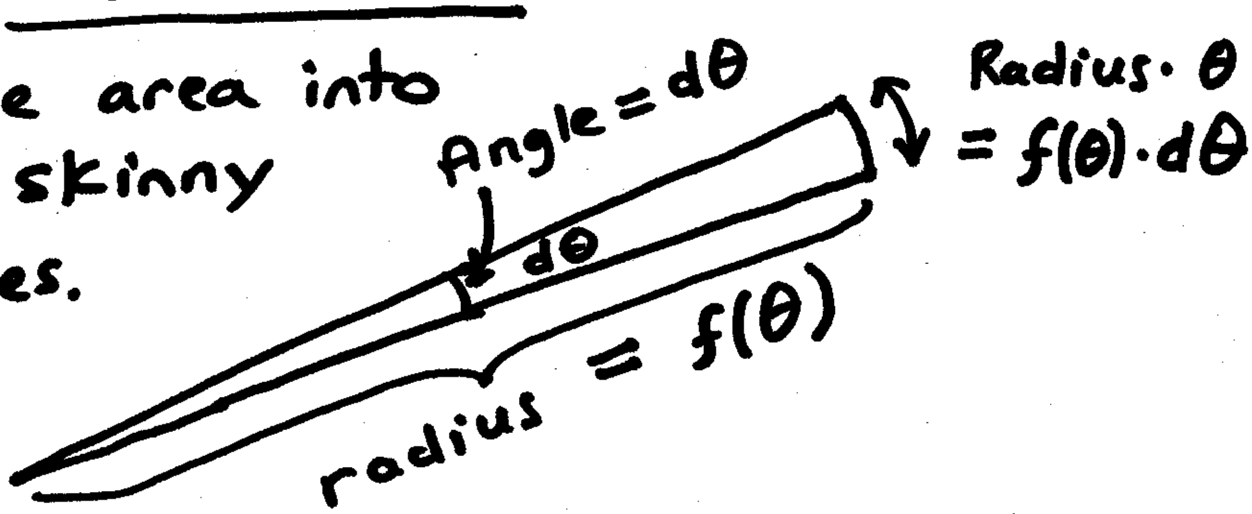
- Assume curve is of the form $r = f(\theta)$.

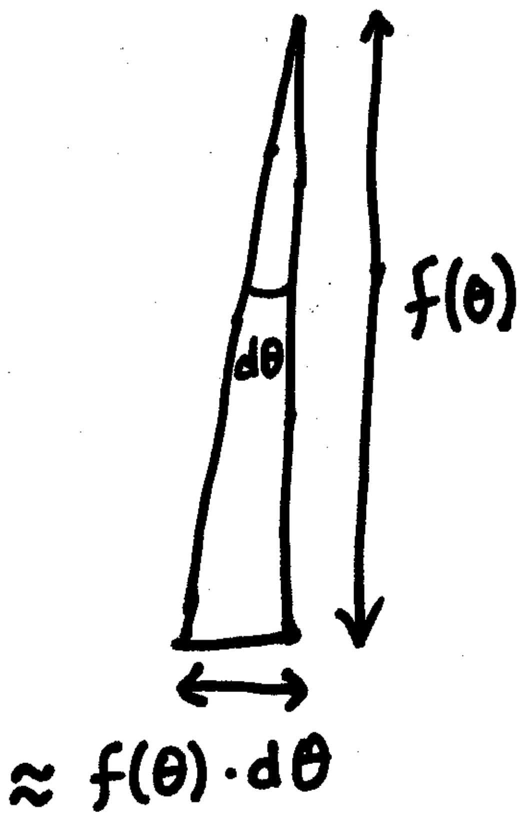


$$\boxed{\text{Area} = \frac{1}{2}\int_a^b f(\theta)^2 \, d\theta.}$$

<u>Why $\frac{1}{2} f(\theta)^2 d\theta$?</u>

Divide area into long, skinny wedges.





This area is very close to a triangle.

Area $= \frac{1}{2}$ (base)(height)

Area of Wedge $\approx \frac{1}{2} f(\theta)\cdot f(\theta)\cdot d\theta$

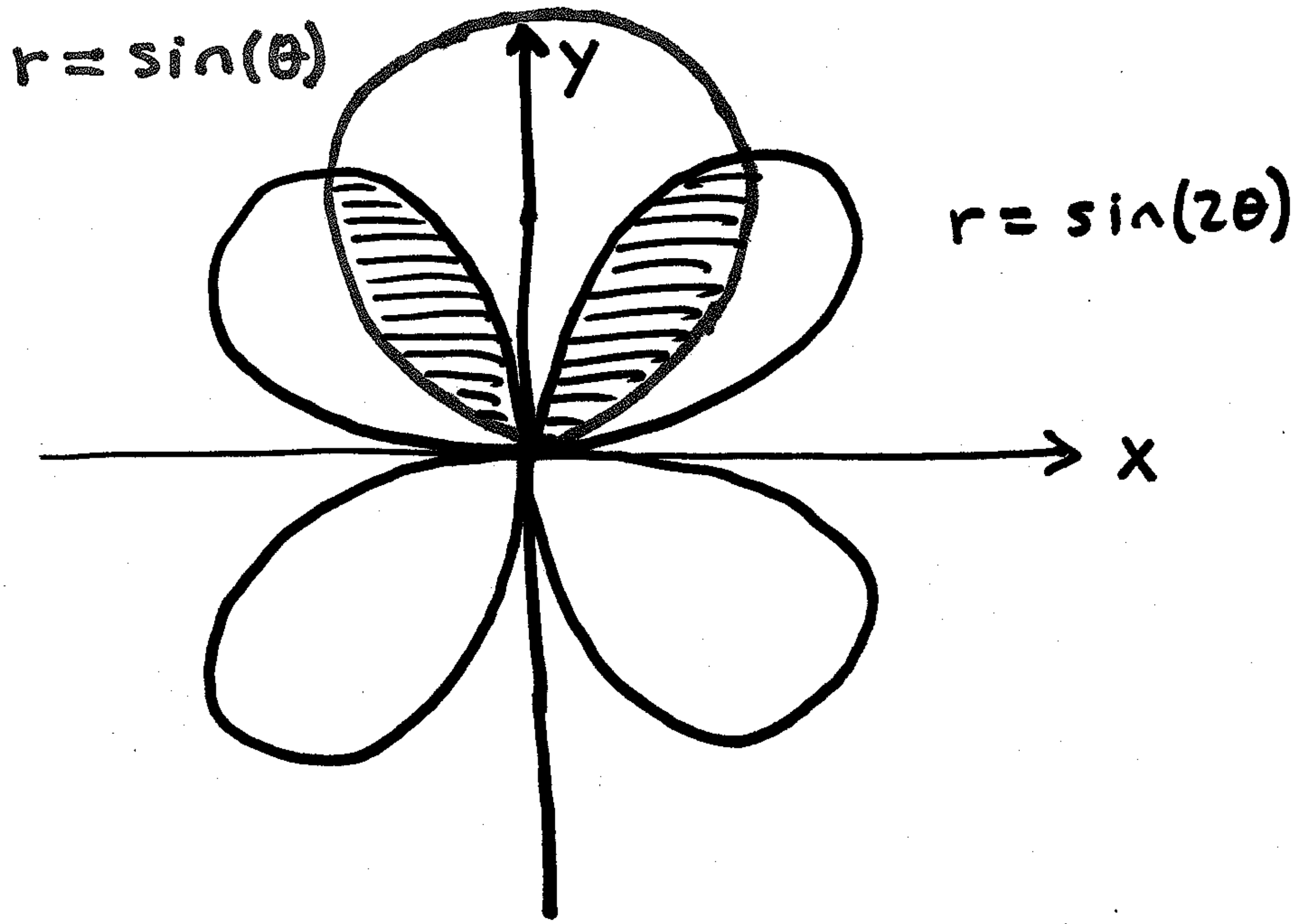
Example

Find the area between:

$r = \sin(2\theta)$ $\qquad$ $r = \sin(\theta)$.

Solution

ALWAYS begin by graphing the curves to find the limits of integration.



Find intersection points:

$$\sin(\theta) = \sin(2\theta)$$

$$\sin(\theta) = 2\sin(\theta)\cos(\theta)$$

$$\sin(\theta) \cdot (1 - 2 \cdot \cos(\theta)) = 0.$$

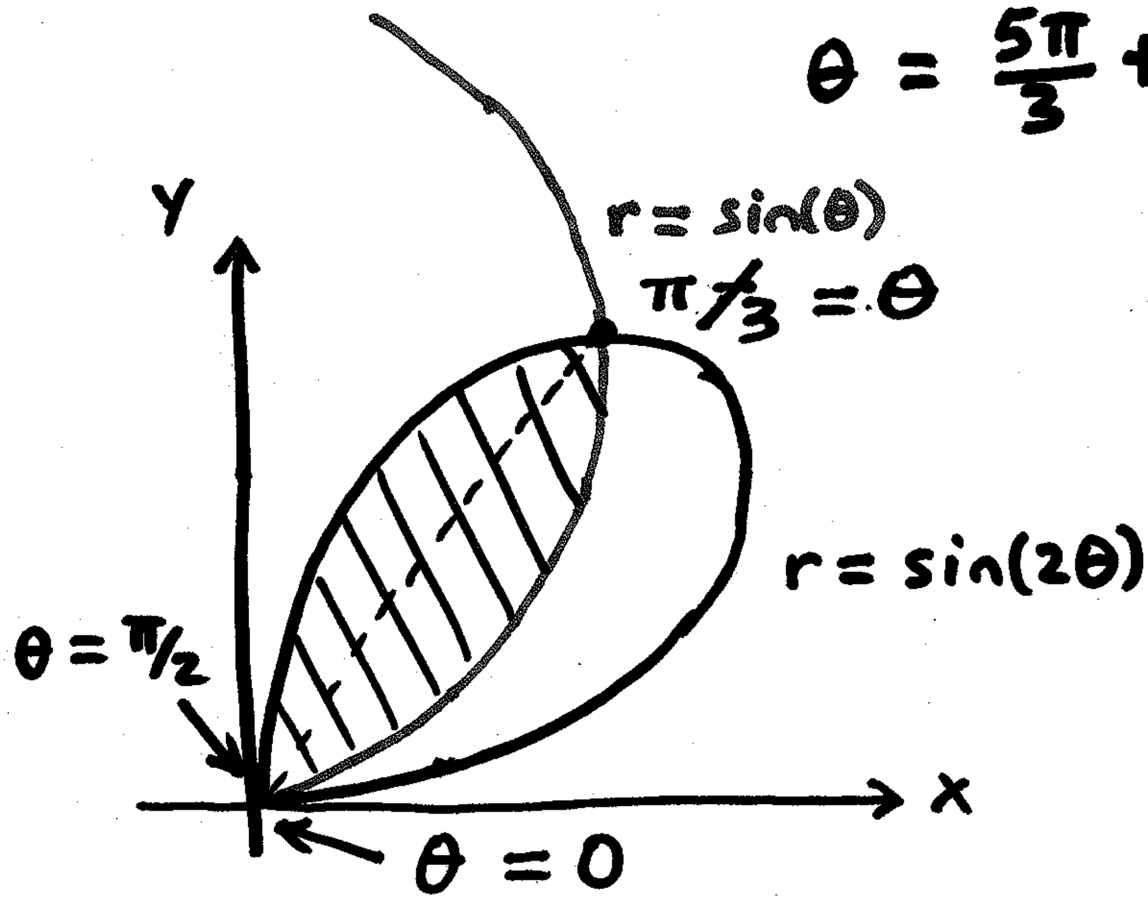
Intersection points:

$\sin(\theta) = 0$ $\qquad$ $\theta = n\pi$ $\quad$ n = integer

$n \in \mathbb{Z}$

$$1 - 2\cdot\cos(\theta) = 0$$

$$\cos(\theta) = \frac{1}{2} \qquad \left.\begin{aligned} \theta &= \frac{\pi}{3} + 2n\pi \\ \theta &= \frac{5\pi}{3} + 2n\pi \end{aligned}\right\} n \in \mathbb{Z}$$



Area above $= \frac{1}{2}\int_0^{\pi/3} \sin^2(\theta)\,d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} \sin^2(2\theta)\,d\theta$

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$

$$= \frac{1}{2}\int_0^{\pi/3} \frac{1}{2}\left(1 - \cos(2\theta)\right)d\theta$$

$$+ \frac{1}{2}\int_{\pi/3}^{\pi/2} \frac{1}{2}\left(1 - \cos(4\theta)\right)d\theta$$

$$= \frac{4\pi - 3\sqrt{3}}{32}$$

Total area enclosed by $r = \sin(\theta)$ and $r = \sin(2\theta)$ is: $\frac{4\pi - 3\sqrt{3}}{16}$.

3. Arc Length with Polar Curves.

- Assume $r = f(\theta)$, then arc length between $\theta = a$ and $\theta = b$ is:

$$\text{Arc length} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

Example

Find the length of:

$$r = \theta^2$$

between $\theta = 0$ and $\theta = 2\pi$.

Solution

$$\text{Arc length} = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$= \int_0^{2\pi} \theta \sqrt{\theta^2 + 4}\, d\theta$$

$u = \theta^2 + 4$

$$= \int_4^{4\pi^2+4} \frac{1}{2} u^{1/2}\, du$$

$$= \left[\frac{1}{3} u^{3/2} \right]_4^{4\pi^2+4}$$

$$= \frac{1}{3}\left((4\pi^2+4)^{3/2} - 4^{3/2} \right).$$