Outline

1. Surface integrals.
2. Stoke's Theorem.
3. The Divergence Theorem.

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Do-over: Thursday 8–9pm, 9–10pm
2210 DH

Final: Friday, May 8 5:30pm
McConomy.

1. Surface Integrals

Let S be a surface with parametric equation $\vec{r}(u,v)$.

assume $a \leq u \leq b$, $c \leq v \leq d$

Write: $\dfrac{\partial \vec{r}}{\partial u}$ as $\vec{r}_u$

$\dfrac{\partial \vec{r}}{\partial v}$ as $\vec{r}_v$

- If $f(x,y,z)$ is a function, the surface integral of f over S is:

$$\iint_S f \, dS = \int_c^d \int_a^b f(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| du\,dv$$

name of surface

always dS

includes special factors like the 'r' in $r\,dr\,d\theta$.

Example

Calculate $\iint_S yz\, dS$ where S

is the surface given by:

$$\vec{r}(u,v) = \langle u^2, u\cdot\sin(v), u\cdot\cos(v) \rangle$$

$$0 \le u \le 1 \quad \text{and} \quad 0 \le v \le \frac{\pi}{2}.$$

Solution

$$f(x,y,z) = y\cdot z$$

$$f(\vec{r}(u,v)) = (u\cdot\sin(v))\cdot(u\cdot\cos(v))$$

$$= u^2\cdot\sin(v)\cdot\cos(v)$$

$$\vec{r}_u = \langle 2u, \sin(v), \cos(v) \rangle$$

$$\vec{r}_v = \langle 0, u\cdot\cos(v), -u\cdot\sin(v) \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -u, -2u^2\sin(v), 2u^2\cos(v) \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 + 4u^4(\sin^2(v) + \cos^2(v))}$$

$$= \sqrt{u^2 + 4u^4}$$

$$\iint_S f\, dS = \int_0^1 \int_0^{\frac{\pi}{2}} u^2 \sin(v)\cos(v)\sqrt{u^2+4u^4}\, dv\, du$$

$$= \frac{5\sqrt{5}}{48} + \frac{1}{240}.$$

- Next, if $\vec{F}(x,y,z)$ is a vector field then:

$$\iint_S \vec{F} \bullet d\vec{S} = \int_c^d \int_a^b \vec{F}(\vec{r}(u,v)) \bullet (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

Example

Calculate $\iint_S \vec{F} \bullet d\vec{S}$ where:

$\vec{F}(x,y,z) = \langle x, y, 3 \rangle$

and S is the part of the paraboloid $z = x^2 + y^2$ from $z = 0$ to $z = 3$.

Solution

Use Polar coordinates to parametrize surface.

$$\vec{r}(r,\theta) = \langle r \cdot \cos(\theta), r \cdot \sin(\theta), r^2 \rangle.$$

$$0 \leq r \leq \sqrt{3} \qquad 0 \leq \theta \leq 2\pi.$$

$$\vec{F}(\vec{r}(r,\theta)) = \langle r\cdot\cos(\theta),\ r\cdot\sin(\theta),\ 3 \rangle$$

$$\vec{r}_r = \langle \cos(\theta),\ \sin(\theta),\ 2r \rangle$$

$$\vec{r}_\theta = \langle -r\cdot\sin(\theta),\ r\cdot\cos(\theta),\ 0 \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle$$

$$\vec{F}(\vec{r}(r,\theta)) \bullet (\vec{r}_r \times \vec{r}_\theta)$$

$$= 3r - 2r^3.$$

factor of 'r' for $r\,dr\,d\theta$ already included

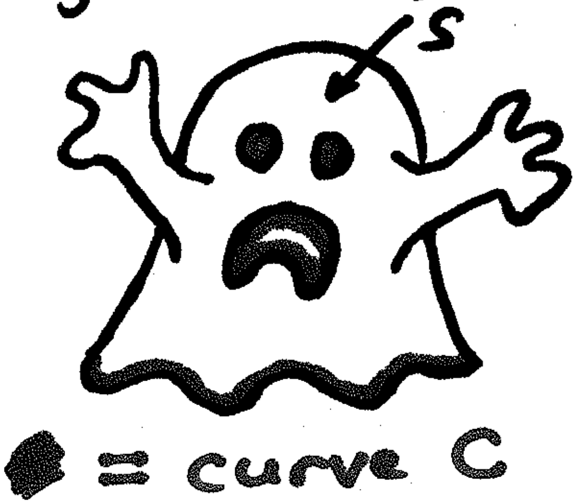
$$\iint_S \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^{\sqrt{3}} (3r - 2r^3)\,dr\,d\theta$$

$$= 2\pi\left(\frac{9}{2} - \frac{1}{2}9\right)$$

$$= 0.$$

2. Stoke's Theorem

Theorem: Let S be a surface with boundary/edge given by a curve C. Let $\vec{F}$



be a vector field whose components have continuous first derivatives on S.

Then: $$\iint_S (\text{curl}(\vec{F})) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$\vec{r}(t)$ gives parametric equations for C.

3. The Divergence Theorem

Theorem: Let S be surface that completely encloses a region of 3D space, D. Let $\vec{F}$ be a vector field whose components have continuous first derivatives.

$$\iint_S \vec{F} \bullet d\vec{S} = \iiint_D \nabla \bullet \vec{F} \, dV.$$

Example

Let S [redacted] be the surface formed by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $z=3$.

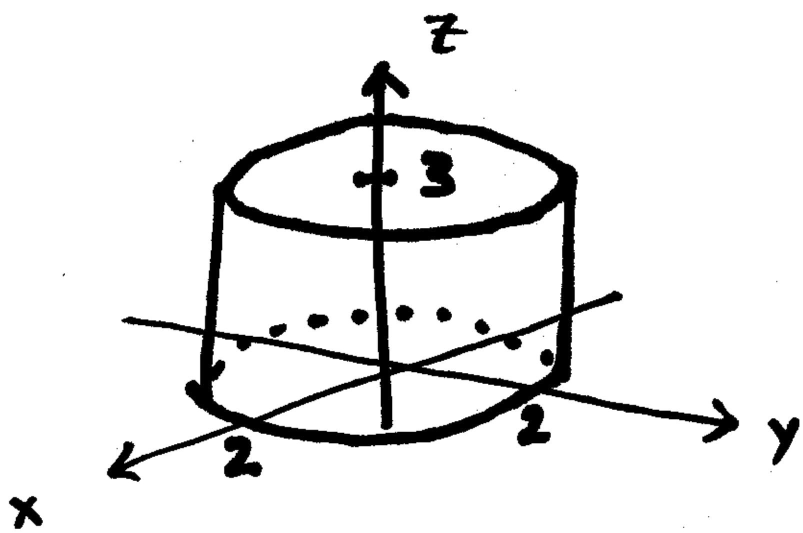
$$\vec{F} = \langle x(x^2+y^2+z^2),\ y(x^2+y^2+z^2),\ z(x^2+y^2+z^2) \rangle$$

Compute: $\iint_S \vec{F} \bullet d\vec{S}$.

Solution:

$$\nabla \bullet \vec{F} = 3x^2 + y^2 + z^2 + x^2 + 3y^2 + z^2$$
$$+ x^2 + y^2 + 3z^2$$
$$= 5(x^2 + y^2 + z^2).$$

Volume:



$$\iint_S \vec{F} \bullet d\vec{S} = \iiint_D \nabla \bullet \vec{F} \, dV$$

$$= \int_0^{2\pi} \int_0^{2} \int_0^{3} 5(r^2+z^2) \cdot r \cdot dz\,dr\,d\theta$$

$$= 300\pi$$