Outline

1. Divergence and curl.
2. Parametric equations for surfaces.
3. Surface integrals.

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1. Quiz (last) on Tuesday.
2. Do-over: Thursday 2210 DH
   8-9pm, 9-10pm.
3. Final: Friday May 8 5:30pm
   Mc Conomy.

1. Divergence and Curl

- Divergence and curl are ways of combining the gradient:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

with vector fields.

- If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ then:

$$\operatorname{div}(\vec{F}) = \nabla \bullet \vec{F}(x,y,z)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$=$ a function of x, y, z.

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}(x,y,z)$$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{from Green's Theorem.}} \right\rangle$$

Example

$\vec{F}(x,y,z) = \langle 2xy,\ x^2 + 2yz,\ y^2 \rangle$.

Compute (a) $\operatorname{div}(\vec{F})$

(b) $\operatorname{curl}(\vec{F})$

(c) $\operatorname{div}(\operatorname{curl}(\vec{F}))$

(d) $\operatorname{curl}(\operatorname{div}(\vec{F}))$.

Solution

(a) $\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2+2yz)$

$$+ \frac{\partial}{\partial z}(y^2)$$

$$= 2y + 2z.$$

(b) $\operatorname{curl}(\vec{F}) = \left| \begin{matrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+2yz & y^2 \end{matrix} \right| \begin{matrix} i & j \\ \partial/\partial x & \partial/\partial y \\ 2xy & x^2+2yz \end{matrix}$

$$= \langle 2y - 2y, 0 - 0, 2x - 2x \rangle$$

$$= \langle 0, 0, 0 \rangle.$$

So $\vec{F}$ is a conservative vector field.

(c) $\operatorname{div}(\operatorname{curl}(\vec{F})) = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z}$

$$= 0.$$

Note: For any vector field $\vec{F}(x,y,z)$,

$$\text{div}(\text{curl}(\vec{F})) = 0$$

provided 2^{nd} partial derivatives of the components of F are continuous.

(d) $\text{curl}(\underbrace{\text{div}(\vec{F})}_{\text{a function, not a vector.}}) = \text{nonsense.}$

can only take the curl of a vector field

2. Parametric Equations for Surfaces

- We want a function that takes 2 numbers as its input and gives a 3D vector as its output. This will be the set of (3) parametric equations that define a surface.

Example

Surface: Hemisphere of radius 5 with center at $(0,0,0)$ and $z \geq 0$.

Find a parametrization of this surface.

Solution

First approach (Cartesian coords):

$$x^2 + y^2 + z^2 = 25, \quad z \geq 0$$

$$z = \sqrt{25 - x^2 - y^2}$$

Parametric equations:

$$\vec{r}(x,y) = \langle x,\ y,\ \sqrt{25-x^2-y^2} \rangle$$

Second approach (Spherical coords):

If radius $= 5$ then $\rho = 5$.

$$x = 5\cos(\theta)\sin(\varphi)$$

$$y = 5\sin(\theta)\sin(\varphi)$$

$$z = 5\cos(\varphi)$$

$$\vec{r}(\theta,\varphi) = \langle 5\cos(\theta)\sin(\varphi),\ 5\sin(\theta)\sin(\varphi),\ 5\cos(\varphi) \rangle$$

$0 \leq \varphi \leq \pi$ to give $z \geq 0$.

Example

Surface: $x^2 + y^2 = 16$

(cylinder).

Find a parametrization.

Solution

Cartesian Approach:

$$y = \sqrt{16 - x^2}$$

$$\vec{r}(x,z) = \langle \underbrace{x, \sqrt{16-x^2}}_{\text{not so great as it requires } y \geq 0.}, z \rangle$$

Polar approach:

$$x = 4\cos(\theta) \qquad y = 4\sin(\theta)$$

$$\vec{r}(\theta, z) = \langle 4\cos(\theta), 4\sin(\theta), z \rangle$$

- Parametric equations for a cylinder: The variable that is not mentioned in the [redacted] formula for the cylindrical surface almost always ends up being one (of the two) variables of the vector function.

3. Surface Integrals