Outline

1. Fundamental Theorem for Line Integrals.
2. Green's Theorem.

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- Recitation in computer cluster Tuesday.
- Recitation in regular room Thursday.

1. Fundamental Theorem for Line Integrals.

Theorem: Let C be a smooth path described by $\vec{r}(t)$, with $a \le t \le b$. Suppose f is a function and ∇f is a continuous vector field. Then:

$$\int_C \nabla f \bullet d\vec{r} = f(\vec{r}(\underset{\substack{\uparrow \\ b}}{b})) - f(\vec{r}(a))$$

Example

Calculate $\int_C \vec{F} \bullet d\vec{r}$ where:

$$\vec{F}(x,y) = \langle x^3y^4,\ x^4y^3 \rangle$$

and $\vec{r}(t) = \langle \sqrt{t},\ 1+t^3 \rangle$

$0 \le t \le 1.$

<u>**Solution**</u>

$\vec{F}(x,y) = \langle x^3y^4,\ x^4y^3 \rangle$ then

$\vec{F}(x,y) = \nabla f(x,y)$ where :

$$f(x,y) = \frac{1}{4}x^4y^4$$

$\vec{r}(0) = \langle 0, 1 \rangle \qquad \vec{r}(1) = \langle 1, 2 \rangle$

By the Fundamental Theorem:

$$\begin{aligned}\int_C \vec{F} \bullet d\vec{r} &= \int_C \nabla f \bullet d\vec{r} \\ &= f(1,2) - f(0,1) \\ &= \frac{1}{4}(1)^4(2)^4 - \frac{1}{4}(0)^4(1)^4 \\ &= 4.\end{aligned}$$

Note: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ then $\vec{F}(x,y)$ is conservative provided:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

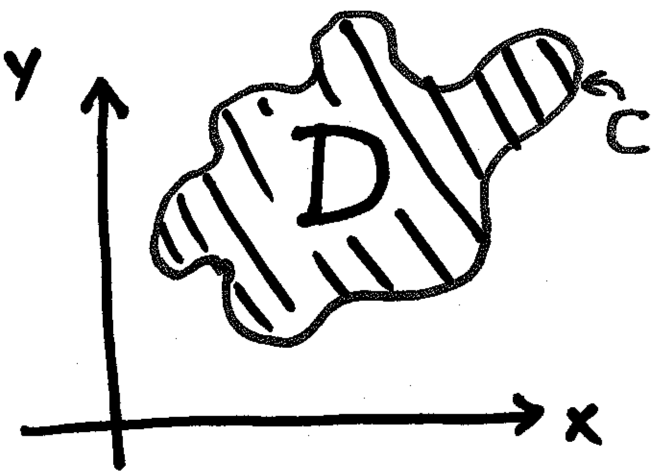
Need this to happen in order to be able to use:

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

2. Green's Theorem

- This converts certain kinds of line integrals into (usually easier) double integrals.

<u>Theorem:</u> Let C be a piecewise smooth closed curve in the xy plane. Let D be the area/region in the xy-plane bounded by C.



If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives at all points in D:

$$\underbrace{\int_C P(x,y)\,dx + Q(x,y)\,dy}_{\text{path integral}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\text{double integral.}}$$

$$\int_C P(x,y)\,dx = \int_a^b P(x(t), y(t)) \cdot x'(t)\,dt$$

$$\int_C Q(x,y)\,dy = \int_a^b Q(x(t), y(t)) \cdot y'(t)\,dt$$

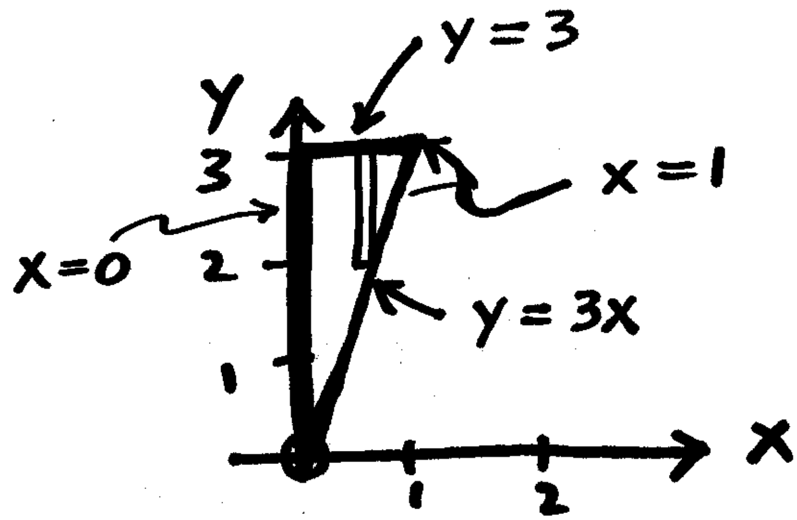
Example

Evaluate: $\int_C x^2y^2\,dx + 4xy^3\,dy$

where C is the edge of the triangle with vertices (0,0), (1,3) and (0,3).

Solution:

Start with sketch of path:



$$P(x,y) = x^2y^2 \qquad \frac{\partial P}{\partial y} = 2x^2y$$

$$Q(x,y) = 4xy^3 \qquad \frac{\partial Q}{\partial x} = 4y^3$$

$$\int_C x^2y^2\,dx + 4xy^3\,dy$$

$$= \int_0^1 \int_{3x}^{3} \left(4y^3 - 2x^2y\right) dy\,dx.$$

$$= \int_0^1 \left[y^4 - x^2y^2 \right]_{3x}^{3} dx$$

$$= \int_0^1 81 - 9x^2 - \left(81x^4 - 9x^4\right) dx$$

$$= \int_0^1 81 - 9x^2 - 72x^4\,dx$$

$$= \left[81x - 3x^3 - \frac{72}{5}x^5 \right]_0^1$$

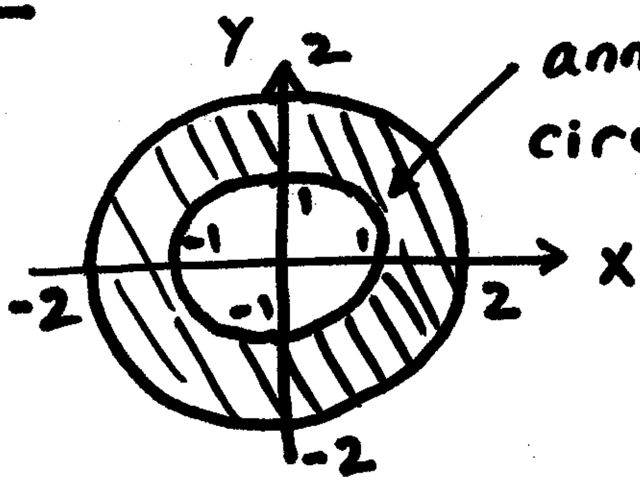
$$= \frac{318}{5}.$$

Example

Calculate $\int_C x e^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$

where C is the boundaries of the two circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution



annuluse between circles is region of integration D.

$$P(x,y) = x e^{-2x} \qquad \frac{\partial P}{\partial y} = 0$$

$$Q(x,y) = x^4 + 2x^2y^2 \qquad \frac{\partial Q}{\partial x} = 4x^3 + 4xy^2$$

$$\int_c xe^{-2x}dx + (x^4+2x^2y^2)dy$$

$$= \int_0^{2\pi}\int_1^2 \left(4r^3\cos^3\theta + 4r^3\cos(\theta)\sin^2(\theta)\right) r\,dr\,d\theta$$

$$= 0.$$