Outline

1. Conservative vector fields.
2. Path integrals of functions.
3. Path integrals of vector fields.

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Next test Friday, April 24.

1. Conservative Vector Fields

- If $\vec{F}(x,y,z)$ is a vector field and $\vec{F}(x,y,z) = \nabla f(x,y,z)$ then $\vec{F}(x,y,z)$ is a conservative vector field.

e.g. $\vec{F}(x,y) = \langle x, y \rangle$

then $\vec{F}(x,y) = \nabla f(x,y)$

where $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ (among many others), so $\vec{F}(x,y)$ is conservative.

e.g. $\vec{F}(x,y) = \langle e^{x^2}, e^{y^2} \rangle$ is not conservative.

- If $\vec{F} = \nabla f$ then f is called a potential function.

2. Path Integrals of Functions

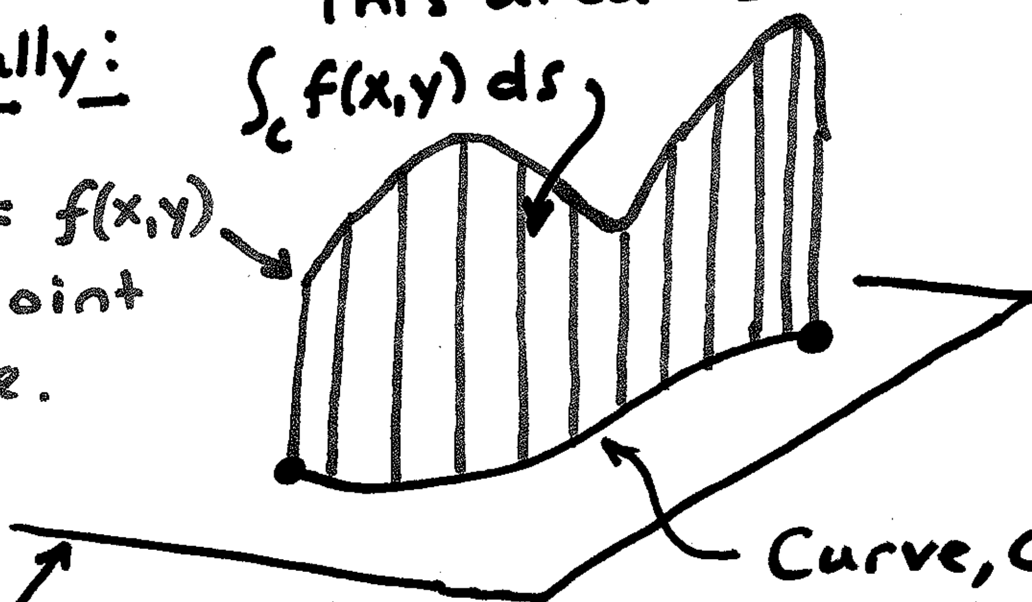
- What is the path integral of a function $f(x,y)$ along a curve C in the xy-plane?

- **Notation:** Path integral $= \int_C f(x,y)\,ds$

  name of curve

  integrate with respect to arc length.

- **Graphically:**



• Evaluating a path integral:

Need: ① function $f(x,y)$

② Curve, C

→ expressed using parametric equations $x(t)$ and $y(t)$

→ interval of t-values $[a,b]$ with $t=a$ corresponding to the starting point of C and $t=b$ corresponding to end point of C.

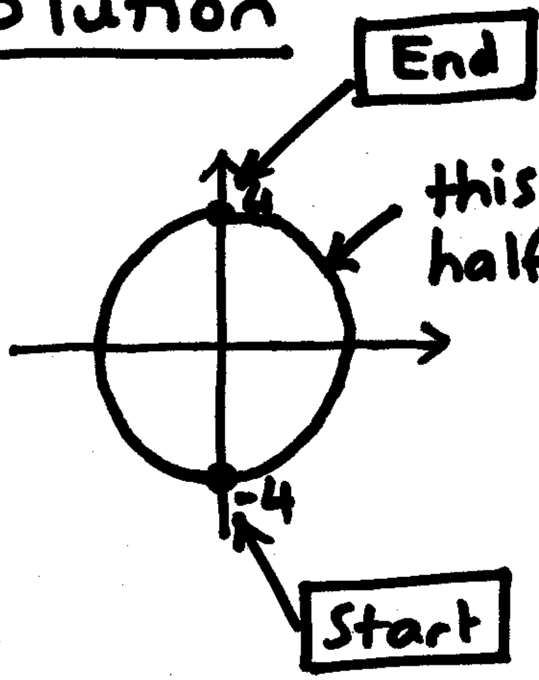
Then:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Example

Calculate $\int_C x y^4 \, ds$ where C is the right half of the circle $x^2 + y^2 = 16$.

Solution



$$x(t) = 4\cos(t)$$

$$y(t) = 4\sin(t)$$

$$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

$$f(x,y) = x\,y^4$$

$$f(x(t), y(t)) = 4 \cdot \cos(t) \cdot \left(4 \cdot \sin(t)\right)^4$$

$$= 4^5 \cdot \cos(t) \cdot \sin^4(t).$$

$$x'(t) = -4\sin(t) \qquad y'(t) = 4\cos(t)$$

$$\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{16 \cdot \sin^2(t) + 16 \cdot \cos^2(t)}$$

$$= \sqrt{16 \cdot (\sin^2(t) + \cos^2(t))}$$

$$= 4.$$

$$\int_C f(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4)^5 \cdot \cos(t) \cdot \sin^4(t) \cdot 4 \; dt$$

$$= \frac{4^6}{5} \cdot \Big[\sin^5(t) \Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{2 \cdot 4^6}{5}$$

- <u>Two Special Path Integrals</u>

- Assume we have $f(x,y)$ and $x(t)$, $y(t)$ and $[a,b]$ describing C.

$$\int_C f(x,y)\,\underbrace{dx}_{} = \int_a^b f(x(t),y(t))\cdot x'(t)\cdot dt$$

note: not ds

derivative of parametric equation for $x(t)$.

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\cdot y'(t)\cdot dt$$

3. Path Integrals of Vector Fields

- Need: ① Vector field $\vec{F}(x,y)$.

  ② Curve C with parametric equations $x(t)$ and $y(t)$ and interval $[a,b]$, put together as:

  $\vec{r}(t) = \langle x(t), y(t)\rangle$.

- To evaluate the path integral of $\vec{F}(x,y)$ along C:

$$\underbrace{\int_C}_{\text{curve, } C} \vec{F} \underbrace{\bullet}_{\text{dot product}} d\vec{r} = \int_a^b \vec{F}\big(x(t), y(t)\big) \bullet \vec{r}\,'(t)\, dt$$

plug parametric equations into vector field

dot product.

<u>Example</u>

Calculate the path integral of $\vec{F}(x,y) = \langle -y, x \rangle$ along the right half of the circle $x^2 + y^2 = 16$.

<u>Solution</u>

$$x(t) = 4\cos(t) \qquad y(t) = 4\sin(t)$$

$$\vec{r}(t) = \langle 4\cos(t),\ 4\sin(t) \rangle$$

$$\vec{r}\,'(t) = \langle -4\sin(t),\ 4\cos(t) \rangle$$

$$\vec{F}(x(t),y(t)) = \langle -y(t),\ x(t) \rangle$$

$$= \langle -4\sin(t),\ 4\cos(t) \rangle$$

$$\vec{F}(x(t),y(t)) \bullet \vec{r}\,'(t) = 16\sin^2(t) + 16\cos^2(t)$$

$$= 16$$

$$\int_C \vec{F} \bullet d\vec{r} = \int_{-\pi/2}^{\pi/2} 16\, dt = 16\pi.$$