Outline

1. Spherical integral.
2. Vector fields.
3. Line integrals.

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- Recitation on Tuesday in computer cluster.
- Last HW due at start of recitation.

1. Spherical Integral

Example

Find the volume bounded by:

- $\rho = 4\cos(\varphi)$
- $\varphi = \pi/3$.

Solution

- Begin by visualizing the 3D shape.

(i) Equation $\boxed{\rho = 4\cos(\varphi)}$

Convert to x, y, z in much the same way as we converted polar curves to Cartesian curves.

$$\rho^2 = 4\rho\cos(\varphi)$$

$$x^2 + y^2 + z^2 = 4z$$

$$x^2 + y^2 + z^2 - 4z + 4 = 4$$

$$x^2 + y^2 + (z-2)^2 = 4$$

- Sphere, radius 2, center at $(0,0,2)$.

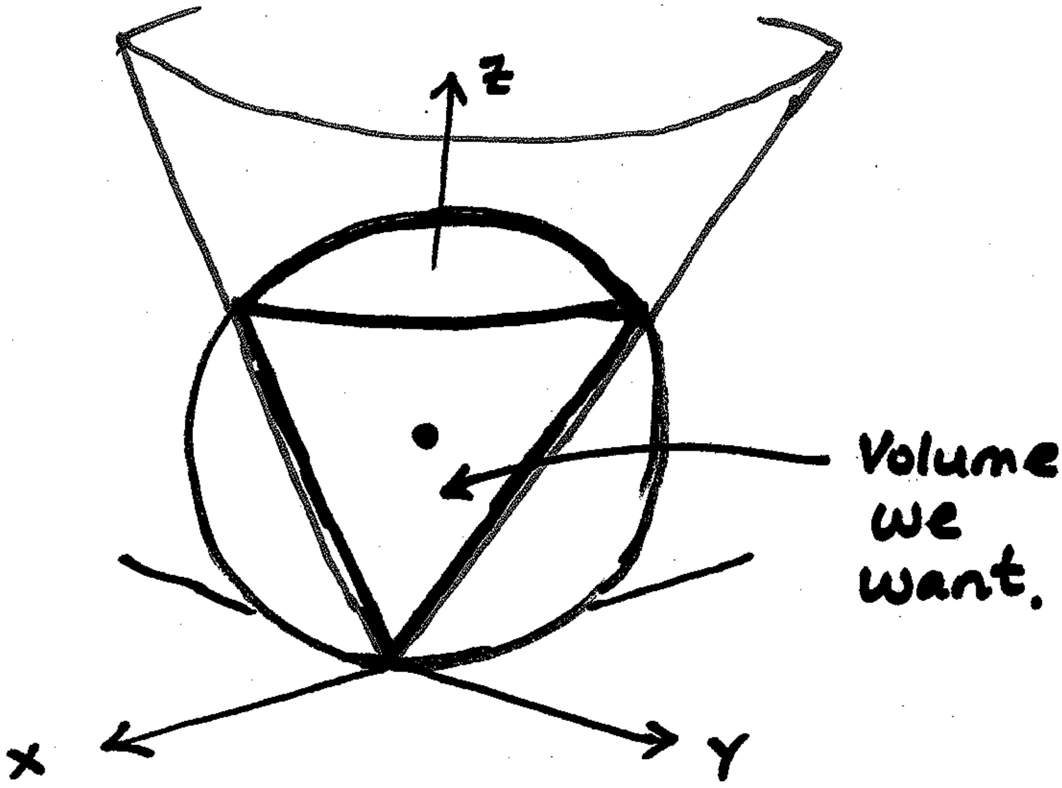
(ii) Equation $\boxed{\varphi = \pi/3}$

- $\underbrace{\rho \geq 0}$ • $\underbrace{0 \leq \theta \leq 2\pi}$

these variables are not mentioned in the equation $\varphi = \pi/3$.

- Cone that opens up the positive z-axis and makes an angle $\pi/3$ with the z-axis.

(iii) Draw the 3D Shape:



$$\text{Volume} = \int_0^{4\cos(\varphi)} \int_0^{\frac{\pi}{3}} \int_0^{2\pi} 1\,\rho^2 \sin(\varphi)\, \underbrace{d\theta\, d\varphi\, d\rho}$$

order is not feasible – $d\rho$ has to come before $d\varphi$ because limits of integration for $d\rho$ involve φ.

$$= \int_0^{\pi/3} \int_0^{2\pi} \int_0^{4\cos(\varphi)} 1 \cdot \rho^2 \cdot \sin(\varphi) \cdot d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \left[\frac{1}{3} \rho^3 \sin(\varphi) \right]_0^{4\cos(\varphi)} d\theta\, d\varphi$$

$$= \frac{64}{3} \int_0^{\pi/3} \int_0^{2\pi} \sin(\varphi) \cos^3(\varphi) \, d\theta \, d\varphi$$

$$= \frac{64}{3} \int_0^{\pi/3} \Big[\theta \cdot \sin(\varphi) \cdot \cos^3(\varphi) \Big]_0^{2\pi} d\varphi$$

$$= \frac{128\pi}{3} \int_0^{\pi/3} \sin(\varphi) \cos^3(\varphi) \, d\varphi$$

$$= \frac{128\pi}{3} \Big[-\frac{1}{4} \cos^4(\varphi) \Big]_0^{\pi/3}$$

$$= 10\pi.$$

2. <u>Vector Fields</u>

- A 2D vector field is a function that takes (x,y) as its input and gives a 2D vector as its output.

$$\vec{F}(x,y) = \langle -y, x \rangle.$$

- A 3D vector field is a function that takes (x,y,z) as its input and gives a 3D vector as its output.

e.g. $\vec{F}(x,y,z) = \langle -y, x, z \rangle.$

Visualizing a Vector Field.

- To do this:
  1. Draw a grid over the xy-plane.
  2. Evaluate the vector field at each point on the grid.
  3. Draw the output vectors on the grid with their tails at the points where they were evaluated.

Example

$$\vec{F}(x,y) = \langle -y, x \rangle.$$