Outline

1. Triple integrals in cylindrical coordinates.
2. Spherical coordinates.

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(a) Quiz Thursday.

(b) Final exam conflicts.

1. Triple Integrals in Cylindrical Coordinates.

- **Idea:** Introduce polar coordinates (r, θ) to replace Cartesian coordinates (x, y) and make a triple integral easier to evaluate.

Example

Find the mass of the object bounded by:

- $z = \sqrt{x^2 + y^2}$
- $z = 3$

with density $\delta(x, y, z) = z$.

Solution

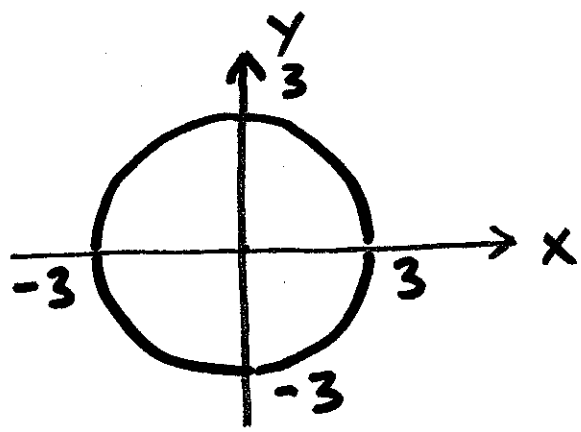
$$\text{Mass} = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} z \, dz \, dy \, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[\frac{1}{2} z^2 \right]_{\sqrt{x^2+y^2}}^{3} dy \, dx$$

$$= \frac{1}{2} \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9 - x^2 - y^2) \, dy \, dx$$

To make this a little easier, convert to polar coordinates.

Region of integration:



- Limits of integration for r: $0 \leq r \leq 3.$
- Limits of integration for θ: $0 \leq \theta \leq 2\pi.$

$$\text{Mass} = \frac{1}{2}\int_0^3 \int_0^{2\pi} (9 - r^2)\, r \cdot d\theta \cdot dr$$

$$= \frac{1}{2}\int_0^3 \Big[9r\theta - r^3\theta \Big]_0^{2\pi} dr$$

$$= \pi \int_0^3 9r - r^3 \, dr$$

$$= \pi \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_0^3$$

$$= \frac{81\pi}{4} \text{ mass units.}$$

- Do the easy integral (dz) first and then convert the rest to polar coordinates.

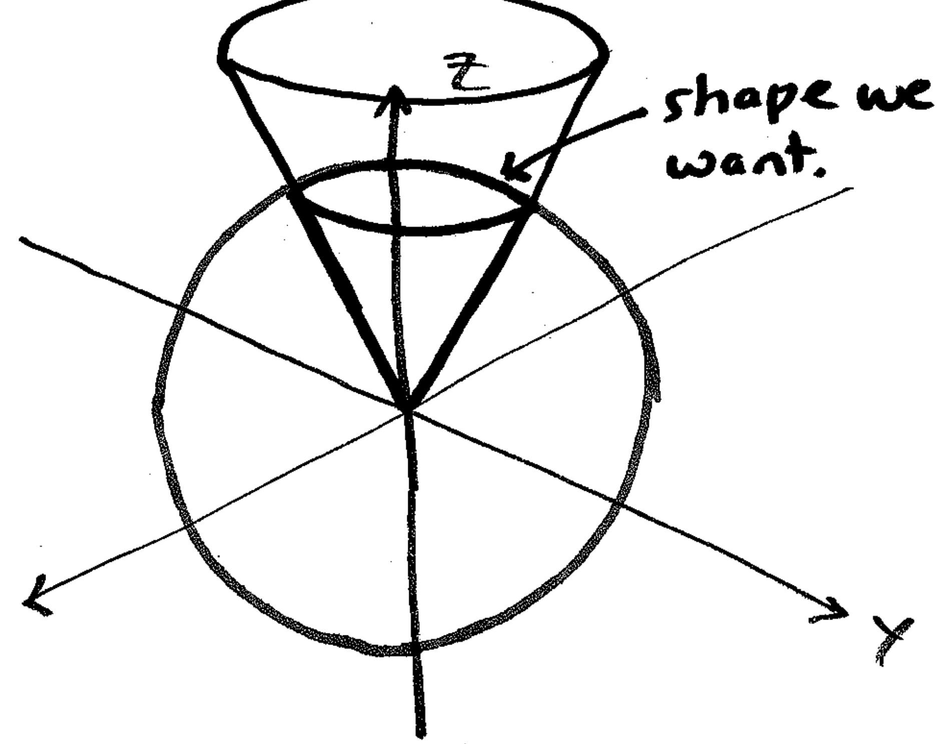
<u>Example</u>

Use a density of $\delta(x, y, z) = z$ to compute the mass of the solid object bounded by:

- $x^2 + y^2 + z^2 = 32$ sphere
- $z = \sqrt{x^2 + y^2}$. cone

Solution

Sketch of 3D region:



Intersection of cone and sphere:

$$x^2 + y^2 + z^2 = 32$$

Square both sides $z = \sqrt{x^2+y^2}$

$$z^2 = x^2+y^2$$

Plug this in to the equation of

the sphere:

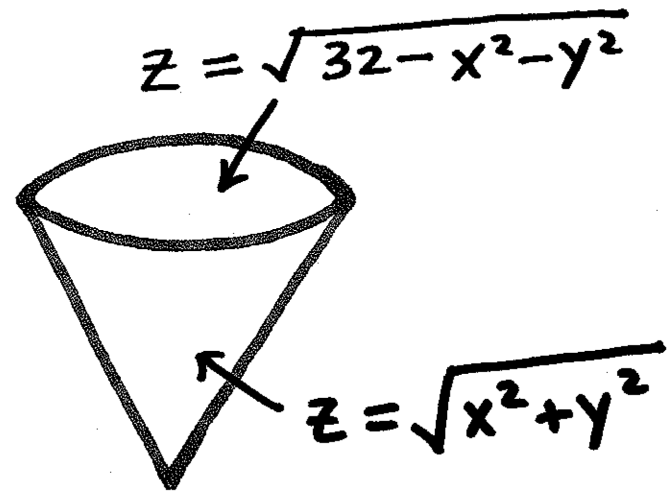
$$x^2 + y^2 + x^2 + y^2 = 32$$

$$x^2 + y^2 = 16$$

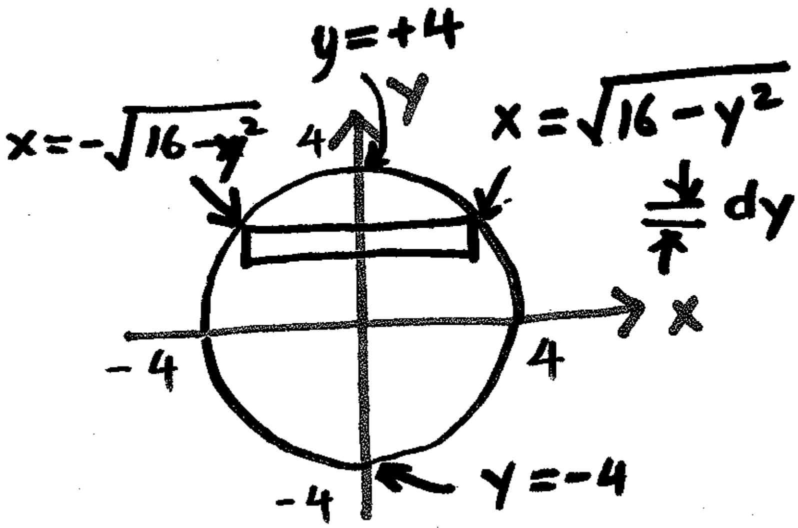
To find z: $\quad z = \sqrt{x^2+y^2} = \sqrt{16} = 4.$

- Isolate one dimension:

  Isolate z.



- Shadow in the xy-plane:



$$\text{Mass} = \int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z \, dz \, dx \, dy$$

$$= \int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \left[\frac{1}{2} z^2 \right]_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} dx\, dy$$

$$= \frac{1}{2} \int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} (32 - 2x^2 - 2y^2)\, dx\, dy$$

- Convert to polar coordinates:

$$= \frac{1}{2} \int_{0}^{4} \int_{0}^{2\pi} (32 - 2r^2)\, r\, d\theta\, dr$$

$$= \frac{1}{2} \int_{0}^{4} \left[32r\theta - 2r^3\theta \right]_{0}^{2\pi} dr$$

$$= \pi \int_{0}^{4} 32r - 2r^3\, dr$$

$$= \pi \left[16r^2 - \frac{1}{2} r^4 \right]_{0}^{4}$$

$$= 128\pi \text{ mass units.}$$