Outline

1. Finish example from last time.

2. Concept of triple integrals.

3. Setting up triple integrals.

— || —

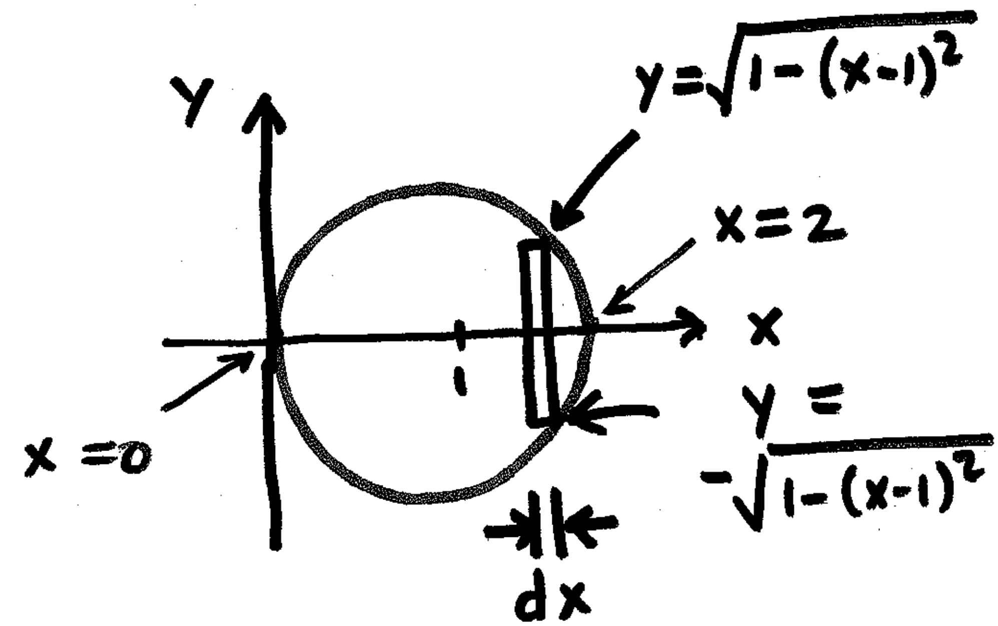
HW due Tuesday at start of recitation.

1. Example from Last Time

Volume enclosed by:

- $x^2 + y^2 + z^2 = 4$
- $(x-1)^2 + y^2 = 1.$

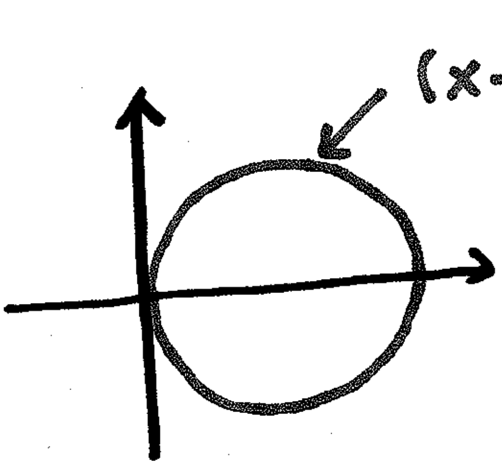
Region of integration:



$$\text{Volume} = \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} 2\sqrt{4-x^2-y^2}\, dy\, dx$$

- To make this easier to compute, convert to r, θ.

Convert Region of Integration:



convert this to polar coords.

$$(x-1)^2 + y^2 = 1$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$x^2 + y^2 = 2x$$

Substitute $x = r\cos(\theta)$

$y = r\sin(\theta)$

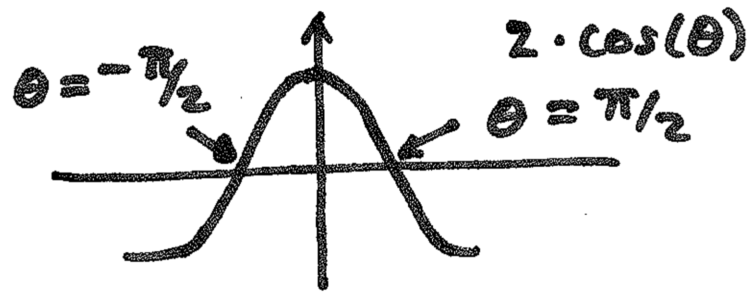
$$r^2 = 2r \cdot \cos(\theta)$$

$$\boxed{r = 2 \cdot \cos(\theta)}$$

Limits of integration for dr integral: $0 \leq r \leq 2 \cdot \cos(\theta)$

What values of θ give $r=0$ when plugged into $r = 2\cos(\theta)$?

$$2\cos(\theta) = 0$$



Limits of integration for $d\theta$ integral: $-\pi/2 \le \theta \le \pi/2$.

Convert Formula:

$$\sqrt{4-(x^2+y^2)} = \sqrt{4-r^2}$$

Integral in r,θ coordinates:

$$\text{Volume} = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos(\theta)} 2\cdot\sqrt{4-r^2}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[2 \cdot \frac{-1}{2} \cdot \frac{2}{3} \left(4 - r^2\right)^{3/2} \right]_0^{2\cos(\theta)} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -\frac{2}{3} \left(\underbrace{4 - 4\cos^2\theta}_{4\sin^2(\theta)}\right)^{3/2} + \frac{2}{3} (4)^{3/2} \, d\theta$$

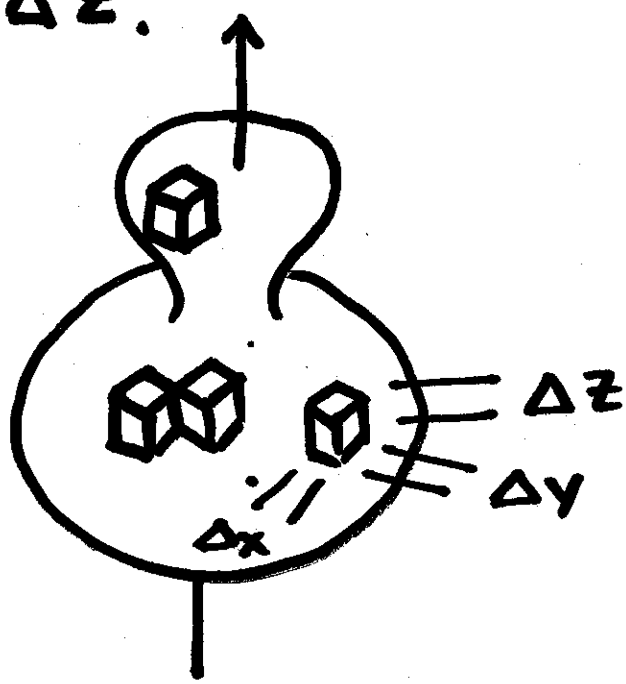
$$= \int_{-\pi/2}^{\pi/2} -\frac{16}{3} \cdot \sin^3(\theta) + \frac{16}{3} \, d\theta$$

$$= \left[\frac{16}{9} \left(2 + \sin^2(\theta)\right)\cos(\theta) + \frac{16\theta}{3} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{16\pi}{3}$$

2. Triple Integrals

- A triple integral arises from dividing a 3D volume into tiny cubes, of side lengths Δx, Δy, Δz.



The total volume is:

$$\sum_{p=1}^{Q} \sum_{j=1}^{M} \sum_{i=1}^{N} 1 \, \Delta x \cdot \Delta y \cdot \Delta z$$

a triple Riemann sum. This

approximation becomes exact in the limit as Δx, Δy, Δz all approach zero.

This limit is written as:

Use 1 when doing volume.

$$\text{Volume} = \int_{z=c}^{z=d} \int_{y=g_1(z)}^{y=g_2(z)} \int_{x=f_1(y,z)}^{x=f_2(y,z)} 1 \, dx\, dy\, dz$$

which is a "triple" integral.

The triple integral of a function $f(x,y,z)$ is written:

$$\int_{z=c}^{z=d} \int_{y=g_1(z)}^{y=g_2(z)} \int_{x=f_1(y,z)}^{x=f_2(y,z)} f(x,y,z)\, dx\, dy\, dz$$

$$\text{Units} = \begin{pmatrix}\text{units}\\ \text{of } f\end{pmatrix}\begin{pmatrix}\text{units}\\ \text{of } x\end{pmatrix}\begin{pmatrix}\text{units}\\ \text{of } y\end{pmatrix}\begin{pmatrix}\text{units}\\ \text{of } z\end{pmatrix}.$$