# Outline

1. Polar coordinates.

2. Converting double integrals from $x, y$ to $r, \theta$.

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Quiz Thursday on double integrals.

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

# 1. Polar Coordinates

$$x = r \cdot \cos(\theta)$$

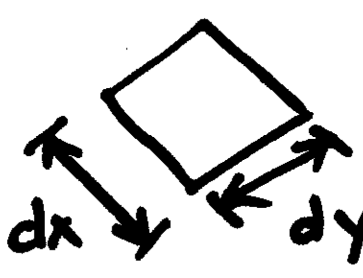
$$y = r \cdot \sin(\theta)$$

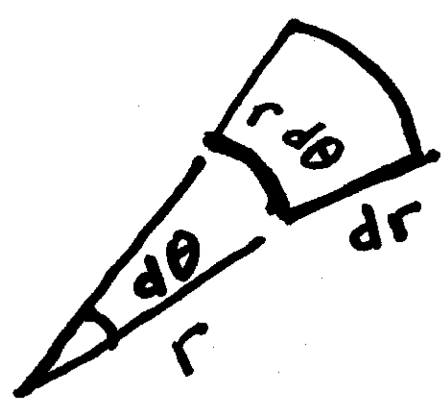
$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$dx\,dy = r\,dr\,d\theta = r\,d\theta\,dr$$

$$dy\,dx = r\,dr\,d\theta = r\,d\theta\,dr.$$




Area ≈ $r\,d\theta\,dr$

# 2. Converting Integrals from $x, y$ to $r, \theta$.

## Example

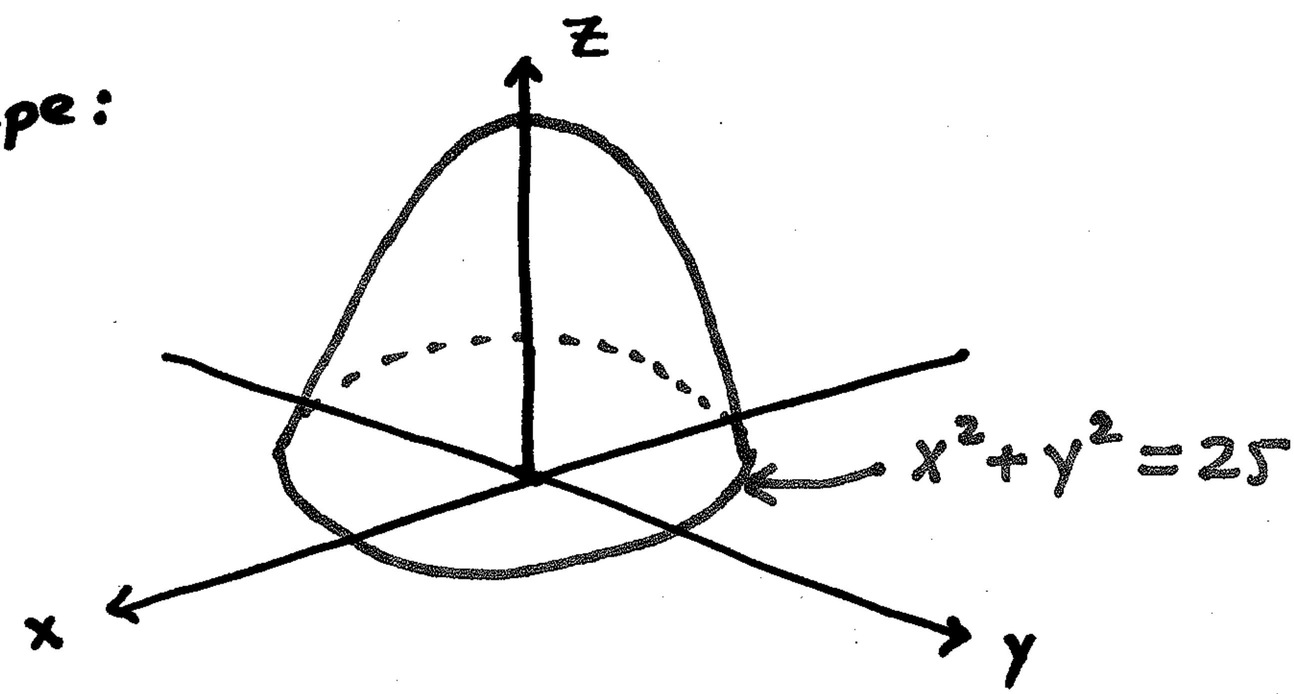
Evaluate the volume between:

$$z = 25 - x^2 - y^2$$

and the xy-plane.

## Solution

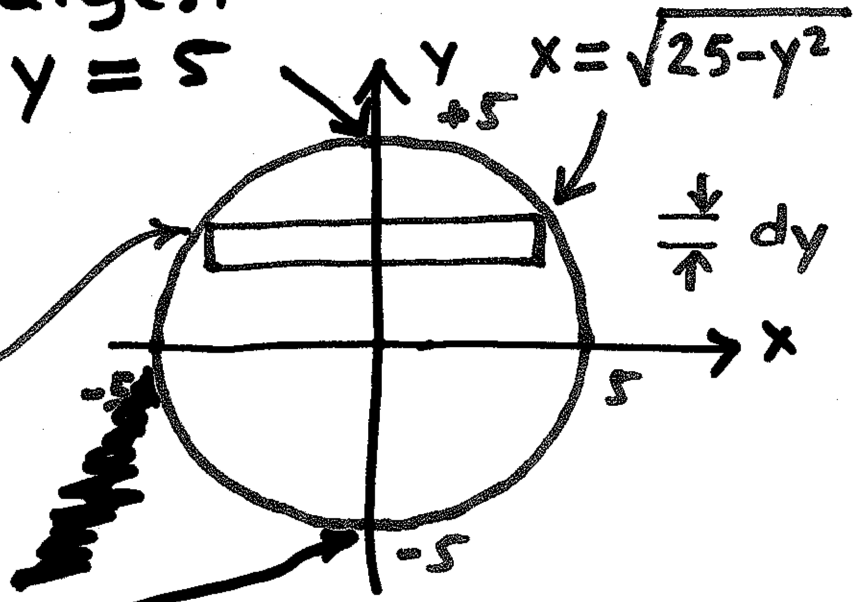
Shape:



Region of integration in xy-plane:

largest $y = 5$

$x = \sqrt{25-y^2}$



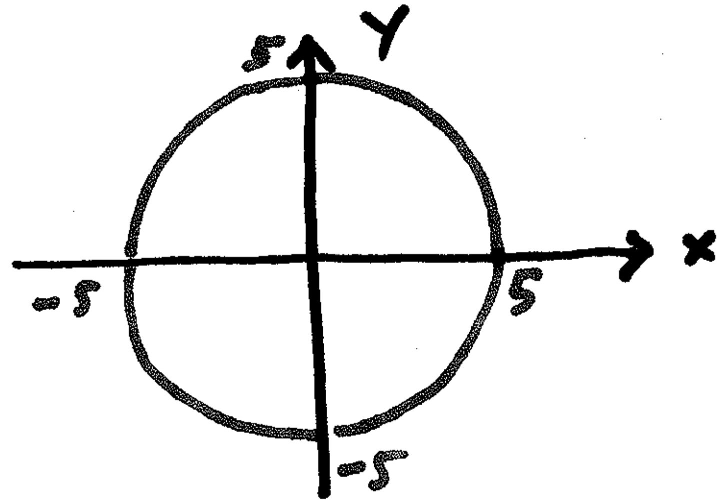

$x = -\sqrt{25-y^2}$

smallest $y = -5$

$$\text{Volume} = \int_{y=-5}^{y=+5} \int_{x=-\sqrt{25-y^2}}^{x=\sqrt{25-y^2}} (25-x^2-y^2)\,dx\,dy$$

- This integral is doable but tough. Easier if we convert to polar coordinates.

Region of integration in terms of $r$ and $\theta$:



The cover the entire disk:

$$0 \leq r \leq 5$$

$$0 \leq \theta \leq 2\pi$$

- These are the new limits of integration.

Had: $f(x,y) = \underbrace{25 - x^2 - y^2}_{25 - r^2}$

- $25 - r^2$ is the function we'll integrate.

- $dx\,dy$ is replaced by $r\,dr\,d\theta$.

$$\underbrace{\int_{-5}^{5}}_{\displaystyle\int_{\theta=0}^{\theta=2\pi}} \underbrace{\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}}}_{\displaystyle\int_{r=0}^{r=5}} \underbrace{(25-x^2-y^2)}_{(25-r^2)} \underbrace{dx\,dy}_{r\,dr\,d\theta}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^5 (25-r^2)r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{25}{2}r^2 - \frac{1}{4}r^4\right]_0^5 d\theta$$

$$= \int_0^{2\pi} \frac{625}{2} - \frac{625}{4}\,d\theta$$

$$= \left[\frac{625}{4}\theta\right]_0^{2\pi}$$

$$= \frac{625\pi}{2}.$$

## When Should I convert?

- If the region of integration is circular in shape.
- If either $x^2+y^2$ or $\sqrt{x^2+y^2}$ appear in the integrand.
- If your integrand cannot be integrated as is, but a factor of $r$ would make it integrable.

no easy anti-deriv.

e.g. $$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy$$

$$= \int_0^{\pi/2} \int_0^1 e^{r^2} \cdot r \, dr\, d\theta$$

u-subst
$u = r^2$

$$= \int_0^{\pi/2} \left[ \frac{1}{2} e^{r^2} \right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}e - \frac{1}{2} \, d\theta$$

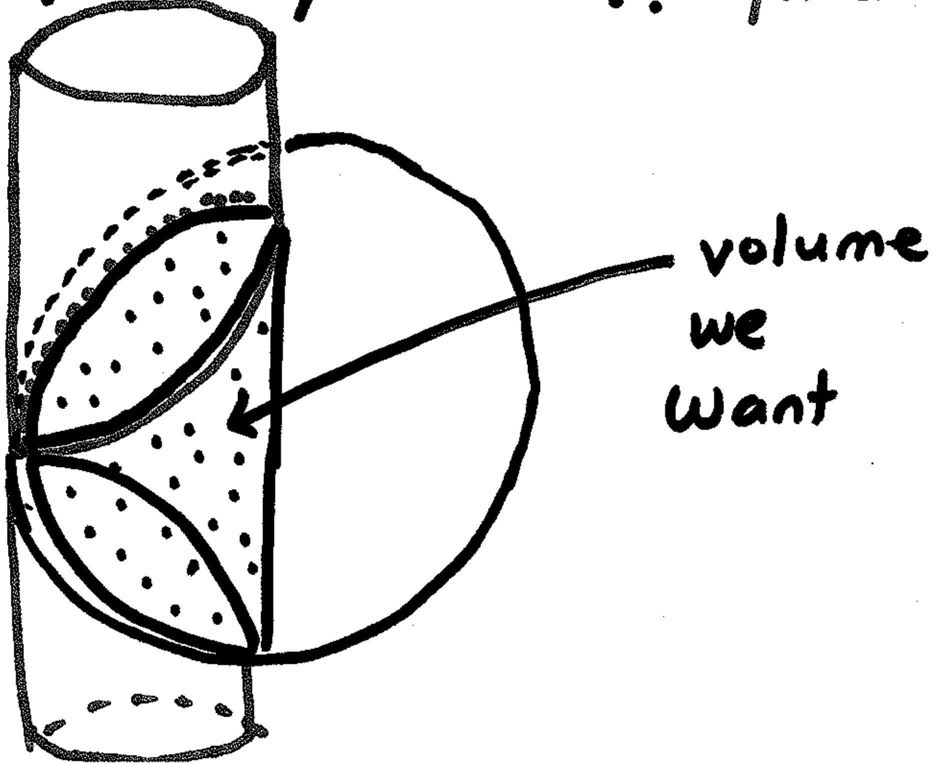
$$= \frac{\pi e}{4} - \frac{\pi}{4}.$$

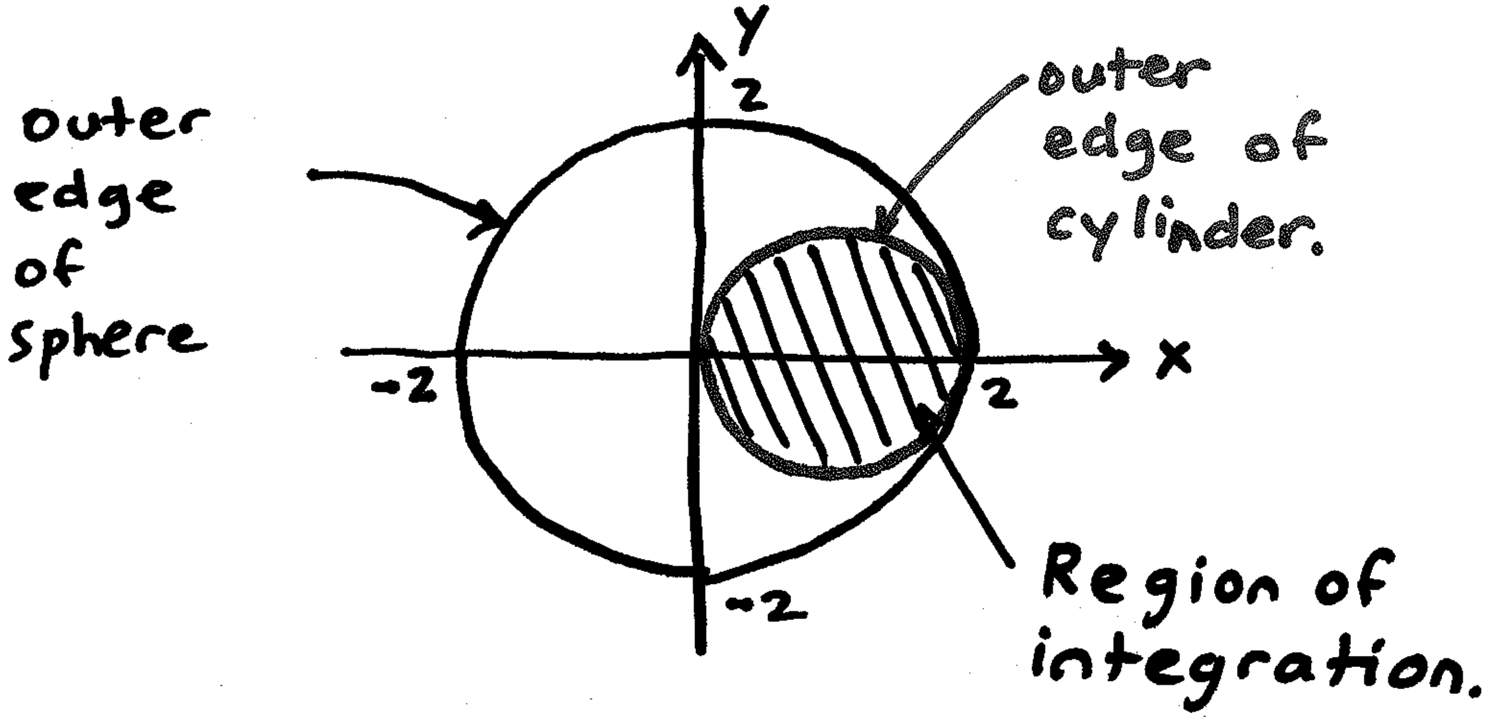
## Example

Find the volume contained within both:

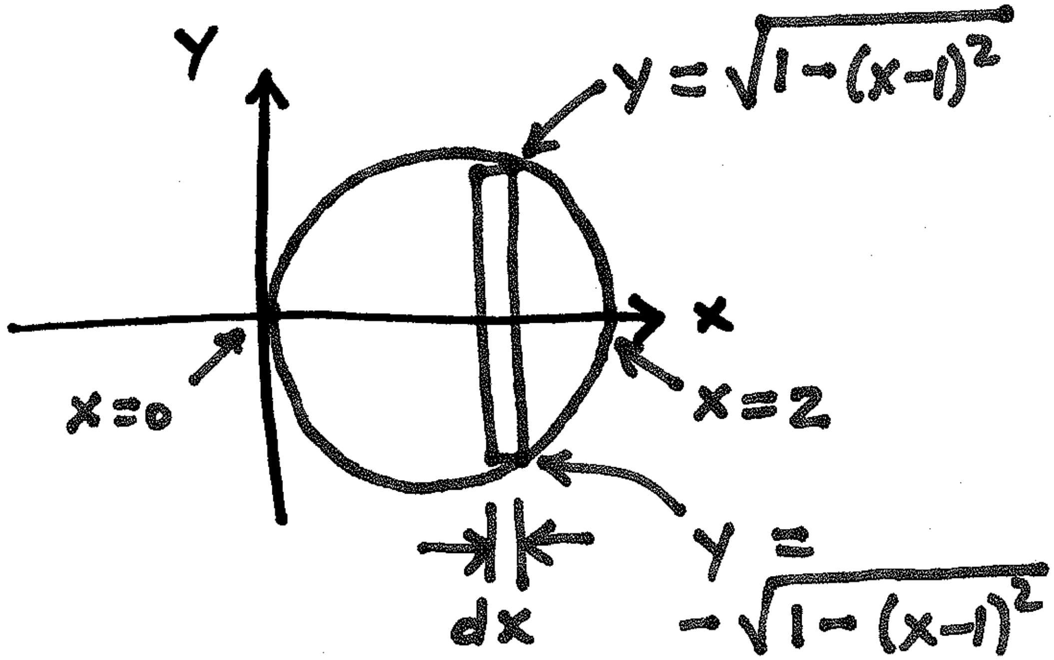
- $x^2 + y^2 + z^2 = 4$ Sphere.
- $(x-1)^2 + y^2 = 1.$ Cylinder.

## Solution






Region of integration




- Height of french fries is given by the sphere $x^2+y^2+z^2=4$.

  Top of french fry: $z=\sqrt{4-x^2-y^2}$

  Bottom of french fry: $z=-\sqrt{4-x^2-y^2}$.

Overall height of french fry $= 2 \cdot \sqrt{4 - x^2 - y^2}$

$$\text{Volume} = \int_{x=0}^{x=2} \int_{y=-\sqrt{1-(x-1)^2}}^{y=\sqrt{1-(x-1)^2}} 2 \cdot \sqrt{4-x^2-y^2} \cdot dy\,dx$$

(to be continued.)