Outline

1. Double integrals over general regions.

2. Changing the order of integration.

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Do-over: Tuesday
8–9 pm and 9–10pm
2210 Doherty.

1. Double Integrals

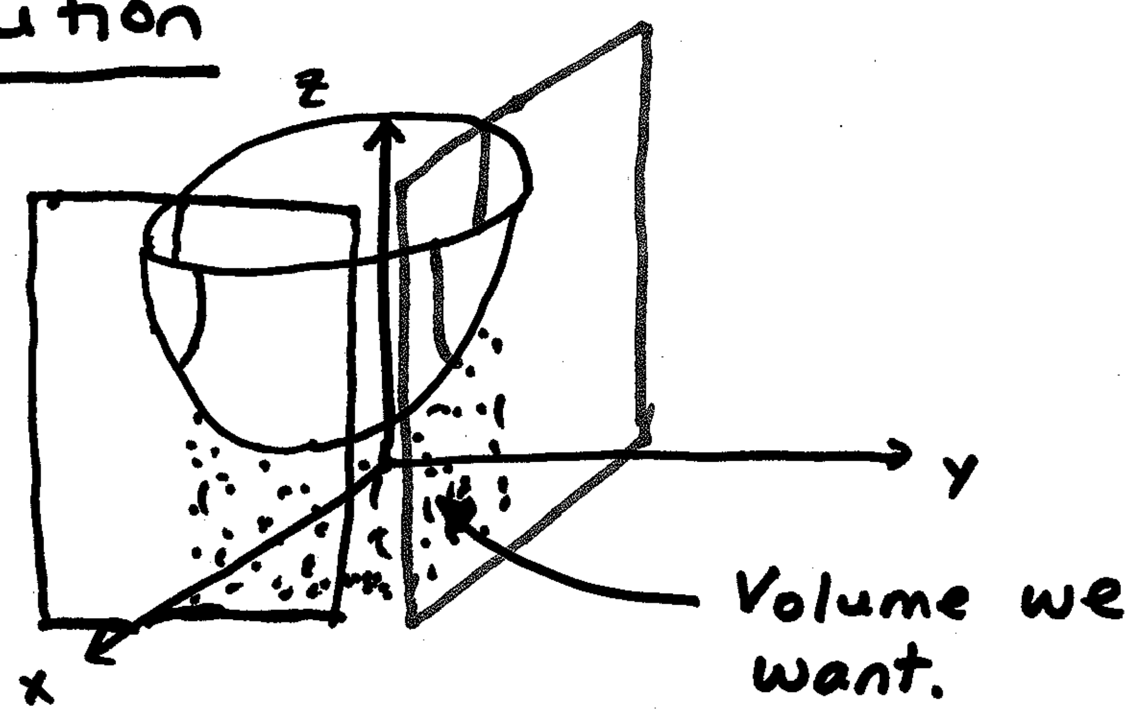
Example

Find the volume enclosed by:

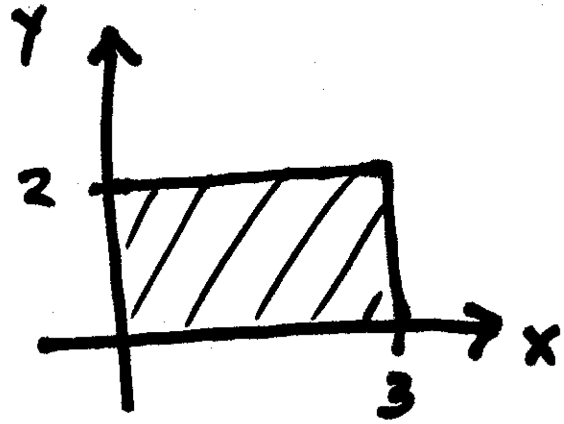
- $z = 1 + (x-1)^2 + 4y^2$
- $x = 3$
- $y = 2$

and the "coordinate planes."

Solution



Region of Integration: ("shadow in xy plane")



French fry volume =

$$(1 + (x-1)^2 + 4y^2) \cdot dx \cdot dy$$

$$\text{Total volume} = \int_{y=0}^{y=2} \int_{x=0}^{x=3} (1 + (x-1)^2 + 4y^2)\, dx\, dy$$

$$= \int_0^2 \left[x + \tfrac{1}{3}(x-1)^3 + 4y^2 x \right]_0^3 dy$$

$$= \int_0^2 3 + \frac{8}{3} + 12y^2 \; dy$$

$$= \left[\frac{17}{3} y + 4y^3 \right]_0^2$$

$$= \frac{34}{3} + 32$$

Double Integrals When the Region of Integration is not a Rectangle

- Double integral is going to look like:

outer limits always numbers

$$\int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\ dy\,dx$$

or

$$\int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\ dx\,dy$$

inner limits may be numbers or functions

Example

Evaluate $\iint_D 2xy\ dA$

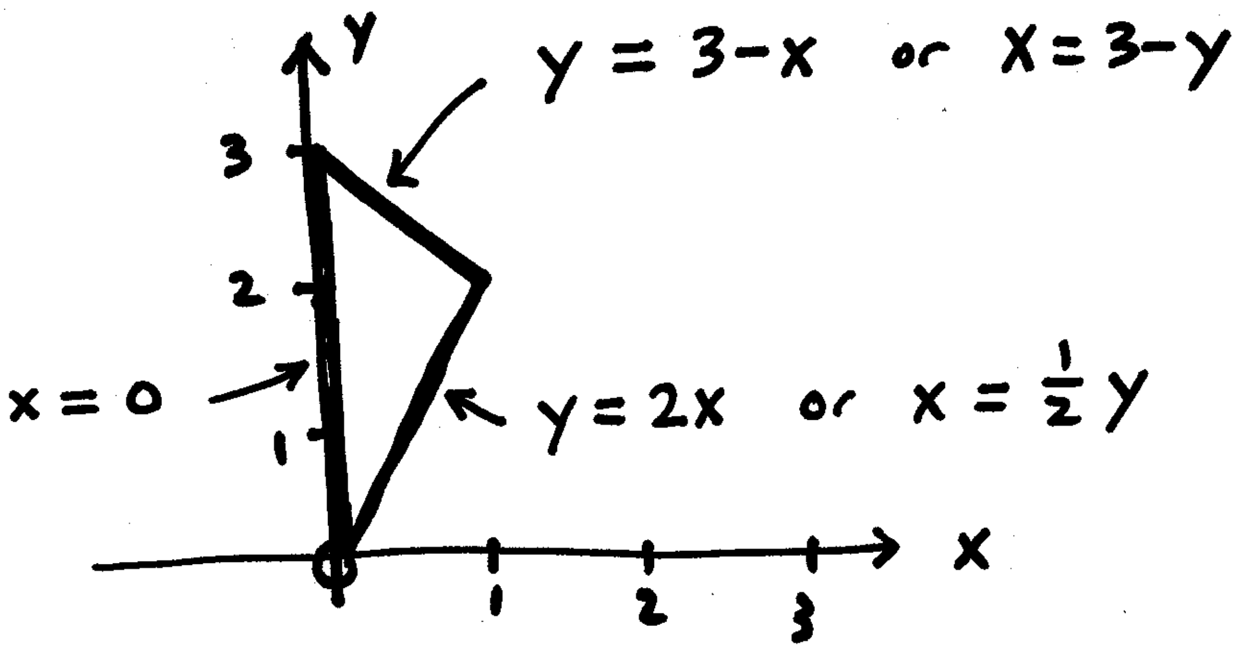
D: some region of the xy-plane (the "shadow")

dA: dxdy or dydx (order up to you)

where D is the triangle with vertices $(0,0)$, $(0,3)$ and [redacted] $(1,2)$.

<u>Solution</u>

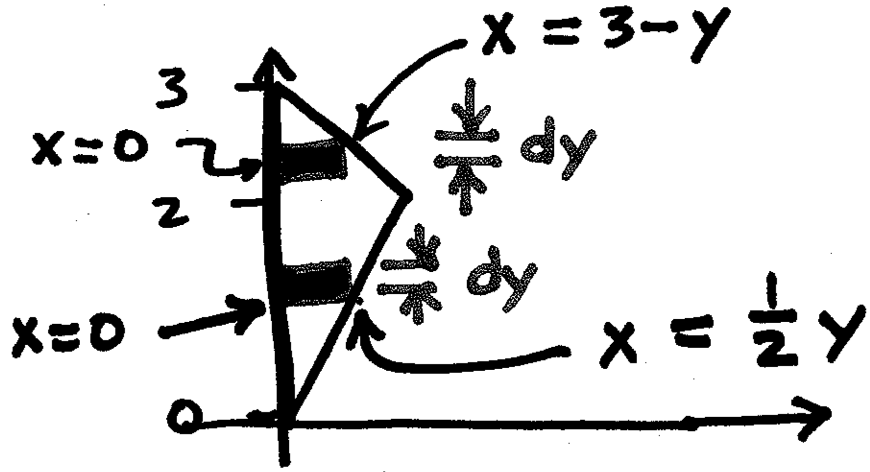
Sketch region of integration:



Set up integral.

First, set it up as a <u>dx dy</u> <u>integral.</u>

- The $\underline{dy}$ integral will be the one done $\underline{\text{last}}$.

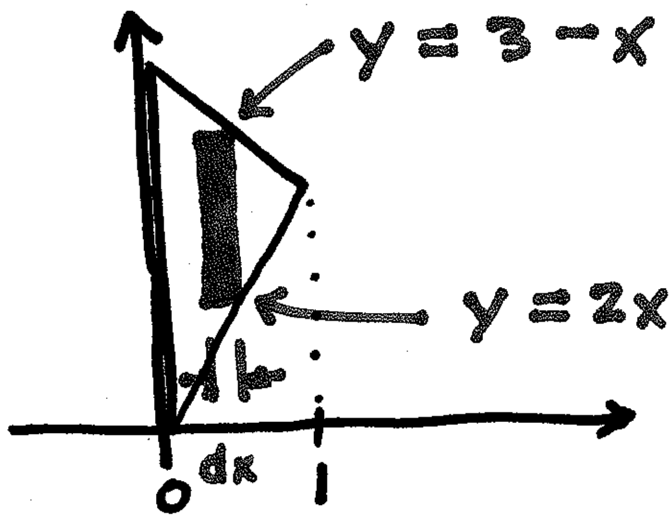


$$\iint_D 2xy\, dA = \int_{y=0}^{y=2} \int_{x=0}^{x=\frac{1}{2}y} 2xy \;\; dx\, dy$$

$$+ \int_{y=2}^{y=3} \int_{x=0}^{x=3-y} 2xy \;\; dx\, dy$$

Set up $\iint_D 2xy\, dA$ as a $\underline{dy\ dx}$ $\underline{\text{integral}}$:

- dx comes last, so slice up the region of integration into rectangles with thickness of dx.



$$\iint_D 2xy \, dA = \int_{x=0}^{x=1} \int_{y=2x}^{y=3-x} 2yx \, dy \, dx.$$

$$= \int_0^1 \left[x y^2 \right]_{2x}^{3-x} dx$$

$$= \int_0^1 x(3-x)^2 - x(2x)^2 \, dx$$

$$= \int_0^1 9x - 3x^3 - 6x^2 \, dx$$

$$= \left[\frac{9}{2}x^2 - \frac{3}{4}x^4 - 2x^3 \right]_0^1$$

$$= 7/4$$

2. Changing the order of integration

e.g.

$$\int_0^1 \int_{x^2}^1 x^3 \cdot \sin(y^3)\, dy\, dx$$

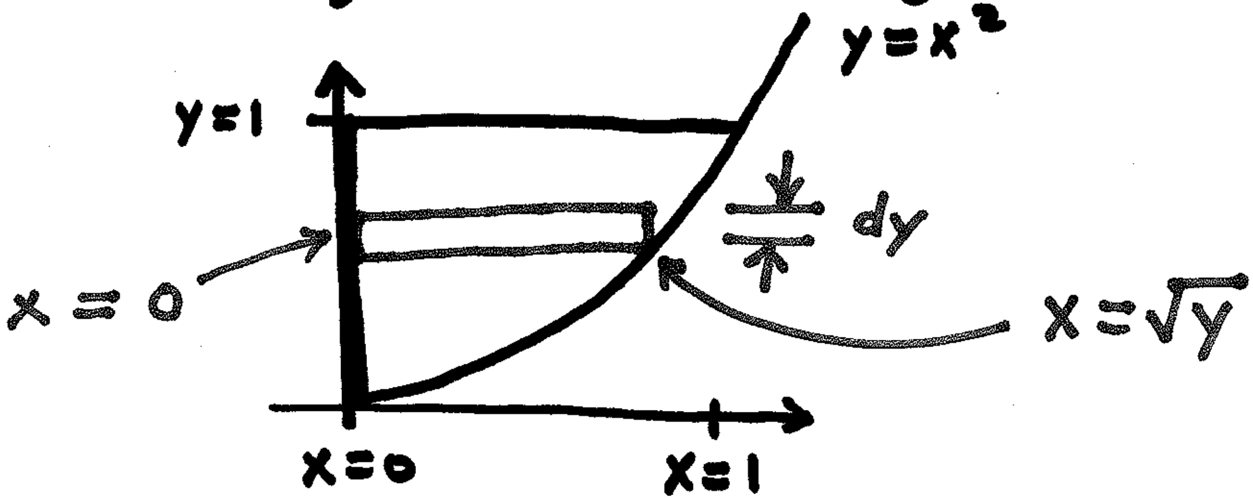
- Can't antidifferentiate $\sin(y^3)$ with respect to y, so change order to integrate with respect to x first.

Example

Evaluate $\int_0^1 \int_{x^2}^1 x^3 \cdot \sin(y^3)\, dy\, dx$.

Solution

Sketch region of integration:



$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$$

$$= \int_{[redacted]0}^{[redacted]1} \int_{x=0}^{x=\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^1 \left[\frac{1}{4} x^4 \cdot \sin(y^3) \right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \frac{1}{4} (\sqrt{y})^4 \cdot \sin(y^3)\, dy$$

$$= \int_0^1 \frac{1}{4} y^2 \cdot \sin(y^3)\, dy \qquad u = y^3$$

$$= \left[\frac{-1}{12} \cos(y^3) \right]_0^1$$

$$= \frac{1}{12} - \frac{\cos(1)}{12}.$$