Outline

1. Approximating volumes.

2. Double integrals.

3. Fubini's Theorem.

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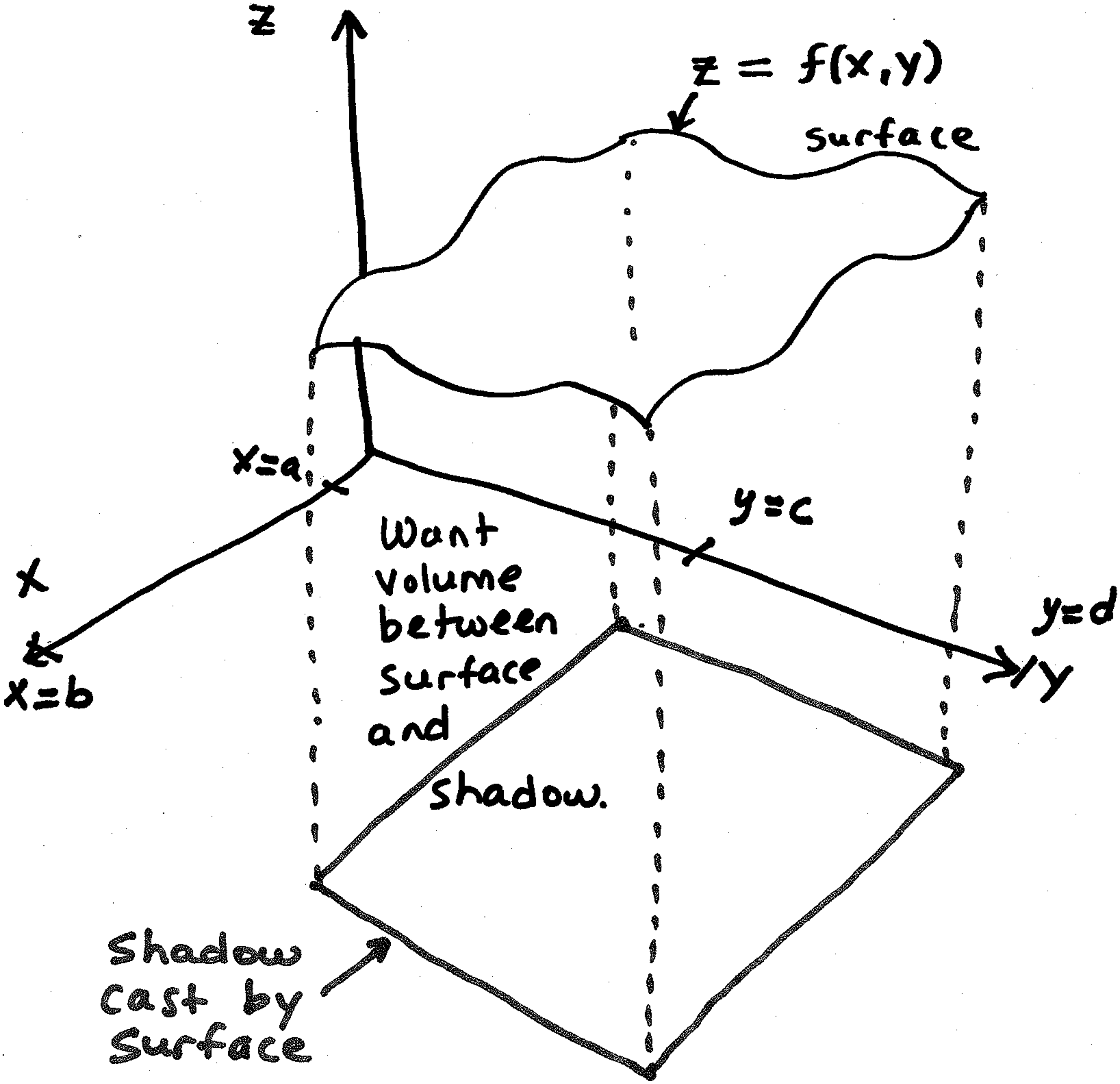
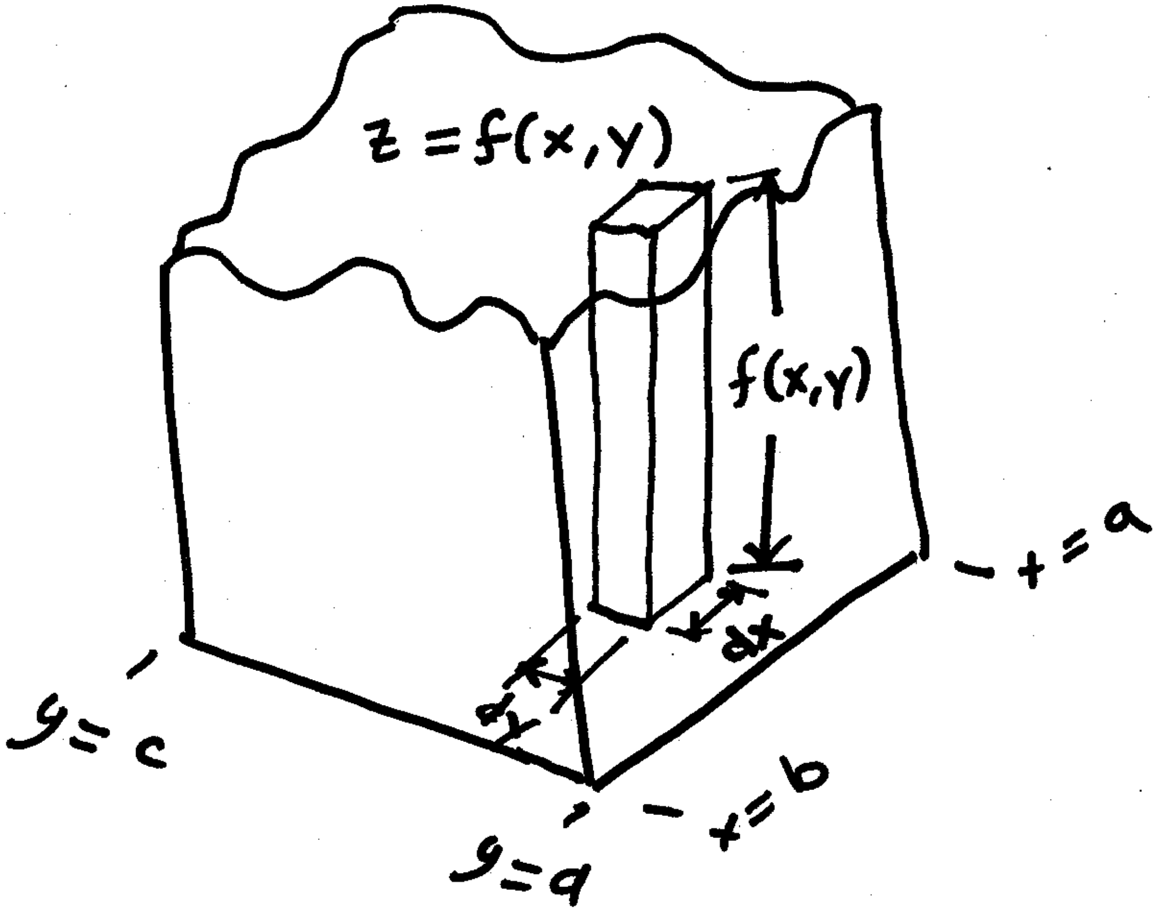
Do-over: Tuesday
8-9 pm, 9-10pm
2210 Doherty.

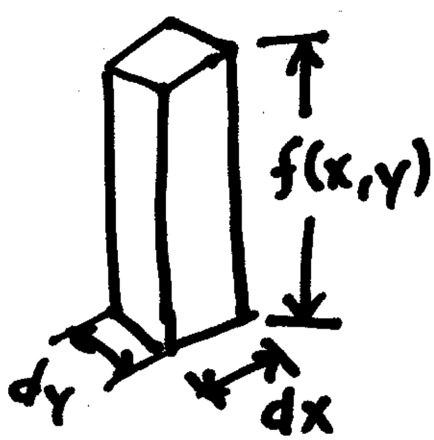
1. Approximating Volume



The volume we want looks like:



- To approximate the volume, we can cut the volume up into french - fry like pieces



Volume of french fry $= f(x,y) \cdot dx \cdot dy$.

- The total volume between $z = f(x,y)$ and the xy-plane is approximately:

$$\text{Volume} \approx \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} f(a + i\cdot\Delta x,\ c + j\cdot\Delta y)\,\Delta x\,\Delta y$$

where: $$\Delta x = \frac{b-a}{N}$$

$$\Delta y = \frac{d-c}{M}.$$

- In the limit as $\Delta x \to 0$ and $\Delta y \to 0$ this approximation becomes exact and:

$$\text{Volume} = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\,dx\,dy$$

- This is a "double" or "multiple" integral.

2. Evaluating Double Integrals

- Integrate one variable at a time, treating other variables as constants.

e.g.

$$\int_{0=y}^{1=y}\int_{2=x}^{3=x} (3x+4y)\,dx\,dy$$

these belong together

these belong together

Example

Evaluate $\int_0^1 \int_2^3 (3x+4y)\,dx\,dy$

Solution

$$\int_0^1 \int_2^3 (3x+4y)\,dx\,dy = \int_0^1 \left[\frac{3}{2}x^2 + 4xy\right]_2^3 dy$$

treat y
as constant

$$= \int_0^1 \frac{3}{2}(3^2) + 4y(3) - \left(\frac{3}{2}(2^2) + 4y(2)\right) dy$$

$$= \int_0^1 \frac{15}{2} + 4y \quad dy$$

$$= \left[\frac{15}{2}y + 2y^2\right]_0^1$$

$$= \frac{15}{2} + 2$$

$$= 19/2.$$

Example

Evaluate $\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx.$

Solution

$u = x^2 \quad u = 1+x^2$

$u = y^2 + x^2 \leftarrow \text{const.}$

$du = 2y\,dy$

$y\,dy = \frac{1}{2}du$

$$\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{1+x^2} \underset{\substack{\uparrow \\ \text{constant.}}}{x} \cdot \frac{1}{2} \cdot \sqrt{u} \cdot du \quad dx$$

$$= \int_0^1 \left[x \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2} \right]_{x^2}^{1+x^2} dx$$

$$= \int_0^1 \frac{1}{3}x\,(1+x^2)^{3/2} - \frac{1}{3}x \cdot (x^2)^{3/2}\,dx$$

$$= \int_0^1 \frac{1}{3}x\,(1+x^2)^{3/2} - \frac{1}{3}x^4\,dx$$

$$u = 1 + x^2 \qquad x=0 \quad u=1$$
$$du = 2x\,dx \qquad x=1 \quad u=2$$
$$\frac{1}{2}du = x\,dx$$

$$= \int_1^2 \frac{1}{3}\cdot\frac{1}{2}\cdot u^{3/2}\,du - \int_0^1 \frac{1}{3}x^4\,dx$$

$$= \left[\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5}u^{5/2}\right]_1^2 - \left[\frac{1}{3}\cdot\frac{1}{5}x^5\right]_0^1$$

$$= \frac{1}{15}(2)^{5/2} - \frac{2}{15}.$$

3. Fubini's Theorem

- So long as $f(x,y)$ is bounded on the rectangle $[a,b]\times[c,d]$,

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$