Outline

1. Lagrange multipliers over a region.

2. Lagrange multipliers with two constraints.

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Do-over: Tuesday 3/31/09
8-9pm <u>or</u> 9-10pm
2210 Doherty.

Drop deadline: Monday 3/30/09.

1. Lagrange Multipliers over a Region.

- Usually the constraint $g(x,y)=0$ is the equation for the boundary/edge of the region.

Example

Find the global min and max:

$$f(x,y) = x^2 + 2y^2$$

over the region:

$$x^2 + y^2 \leq 1.$$

Solution

- Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ to find critical points inside the region.

- Then use Lagrange multipliers with: $f(x,y) = x^2 + 2y^2$

$$g(x,y) = \underbrace{x^2 + y^2 - 1}_{\text{edge of the region.}} = 0.$$

Interior Critical Points

$$\frac{\partial f}{\partial x} = 2x = 0 \qquad \frac{\partial f}{\partial y} = 4y = 0$$

Only solution: $(x,y) = (0,0)$.

Boundary Search

$$\nabla f = \langle 2x, 4y \rangle$$

$$\nabla g = \langle 2x, 2y \rangle.$$

$$\nabla f = \lambda \cdot \nabla g$$

$$2x = 2\lambda x \quad \cdots ①$$
$$4y = 2\lambda y \quad \cdots ②$$

Case 1: $x = 0$

Plug $x = 0$ into constraint:

$$0^2 + y^2 = 1$$
$$y = \pm 1.$$

Case 2: $x \neq 0$

Equation ①: $2 = 2\lambda$

$1 = \lambda$

Equation ②: $4y = 2y$

$y = 0$

Plug $y = 0$ into constraint:

$$x^2 + 0^2 = 1$$
$$x = \pm 1.$$

- To find global max and min of $f(x,y) = x^2 + 2y^2$, make table:

x	y	$f(x,y)$	Comments
0	0	0	Global min of $f(x,y)$
0	1	2	Global max
0	-1	2	of $f(x,y)$.
1	0	1	
-1	0	1	

2. Lagrange Multipliers with more than one constraint.

- If we want the global min and max of $f(x,y,z)$ subject to two constraints:

$$g(x,y,z) = 0$$
$$h(x,y,z) = 0$$

then the system of equations you have to solve is:

$$\nabla f = \lambda_1 \cdot \nabla g + \lambda_2 \cdot \nabla h$$

- Other than the extra $\lambda_2 \nabla h$, everything else is the same.

Example

The plane $x + y + z = 12$ intersects the elliptic paraboloid

$$z = x^2 + y^2$$

in a curve that forms an ellipse. What's the highest point on the ellipse? What's the lowest point on the ellipse?

Solution

$$f(x, y, z) = z.$$

Constraints:

On plane: $g(x, y, z) = x + y + z - 12 = 0.$

On paraboloid: $h(x, y, z) = z - x^2 - y^2 = 0.$

$$\nabla f = \langle 0, 0, 1 \rangle$$

$$\nabla g = \langle 1, 1, 1 \rangle$$

$$\nabla h = \langle -2x, -2y, 1 \rangle$$

$$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$$

$$0 = \lambda_1 - 2\lambda_2 x \quad \ldots (1)$$

$$0 = \lambda_1 - 2\lambda_2 y \quad \ldots (2)$$

$$1 = \lambda_1 + \lambda_2 \quad \ldots (3)$$

$\underline{\lambda_2 = 0:}$ Then (1), (2) ~~imply~~ that $\lambda_1 = 0$.

But $\lambda_1 = \lambda_2 = 0$ does not satisfy (3). So $\lambda_2 \neq 0$.

$\underline{\lambda_2 \neq 0:}$ Rearrange ① and ②:

$$2\lambda_2 x = \lambda_1 = 2\lambda_2 y$$

so $\boxed{x = y}$

Substitute into $h(x,y,z) = 0$

$$x^2 + x^2 = z$$

$$\boxed{z = 2x^2}$$

Substitute into $g(x,y,z) = 0$

$$x + x + 2x^2 = 12$$

$$2x^2 + 2x - 12 = 0$$

Quadratic formula: $x = -3$

$x = 2.$

Points for global min/max:

$(-3, -3, 18)$ $(2, 2, 8)$

- Find global min/max by making table:

x	y	z	$f(x,y,z)$	Comment
-3	-3	18	18	Global max
2	2	8	8	Global min