Outline

1. Lagrange multipliers.
2. Why Lagrange multipliers work.

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Do-over: Tuesday, March 31

8–9 pm <u>or</u> 9–10pm

2210 Doherty.

1. Lagrange Multipliers

- Find the global max and global min of a function $f(x,y)$ which is subject to a constraint: $g(x,y)=0$.

How to do this:

① Calculate ∇f, ∇g.

② Write out the system of equations:

$$\nabla f = \lambda \cdot \nabla g$$

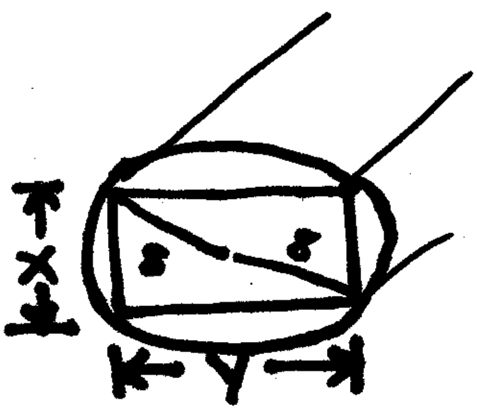
constant, the multiplier.

③ Find all solutions of the system of equations.

④ Evaluate $f(x,y)$ at all of these points. Biggest value of $f(x,y)$ = global max, smallest = global min.

Example

Strength of wooden beam $= 5xy^2$



Beam cut from a log of radius 8 inches.

What should x and y be to maximize strength?

Solution

$$f(x,y) = 5xy^2$$

Constraint: $x^2 + y^2 = 16^2$

$$g(x,y) = x^2 + y^2 - 16^2 = 0.$$

$$\nabla f = \langle 5y^2, 10xy \rangle$$

$$\nabla g = \langle 2x, 2y \rangle$$

System of equations: $\nabla f = \lambda \nabla g$

$$5y^2 = 2\lambda x \quad \ldots ①$$

$$10xy = 2\lambda y \quad \ldots ②$$

Need to find <u>all</u> solutions of ① and ②.

<u>Case 1:</u> $y = 0$

Constraint: $x^2 + 0^2 = 16^2$

$$x = \pm 16.$$

<u>Case 2:</u> $y \neq 0$

$$10x = 2\lambda \quad \ldots ②$$

Subst. into ① :

$$5y^2 = (2\lambda)x$$

$$5y^2 = (10x)x$$

$$5y^2 = 10x^2$$

$$y^2 = 2x^2$$

Substitute into constraint:

$$x^2 + y^2 = 16^2$$

$$x^2 + 2x^2 = 16^2$$

$$x^2 = \frac{16^2}{3}$$

$$x = \pm\sqrt{\frac{16^2}{3}} = \pm\frac{16}{\sqrt{3}}$$

$$y = \pm\sqrt{2}\,x = \pm\frac{16\sqrt{2}}{\sqrt{3}}$$

- **All** solutions to $\nabla f = \lambda \nabla g$ are:

x	y	$f(x,y)$
16	0	0
-16	0	0
$16/\sqrt{3}$	$16\sqrt{2}/\sqrt{3}$	$\dfrac{10 \cdot 16^3}{3\sqrt{3}}$
$16/\sqrt{3}$	$-16\sqrt{2}/\sqrt{3}$	$\dfrac{10 \cdot 16^3}{3\sqrt{3}}$
$-16/\sqrt{3}$	$16\sqrt{2}/\sqrt{3}$	$-\dfrac{10 \cdot 16^3}{3\sqrt{3}}$
$-16/\sqrt{3}$	$-16\sqrt{2}/\sqrt{3}$	$-\dfrac{10 \cdot 16^3}{3\sqrt{3}}$

For realistic global max of strength, use:

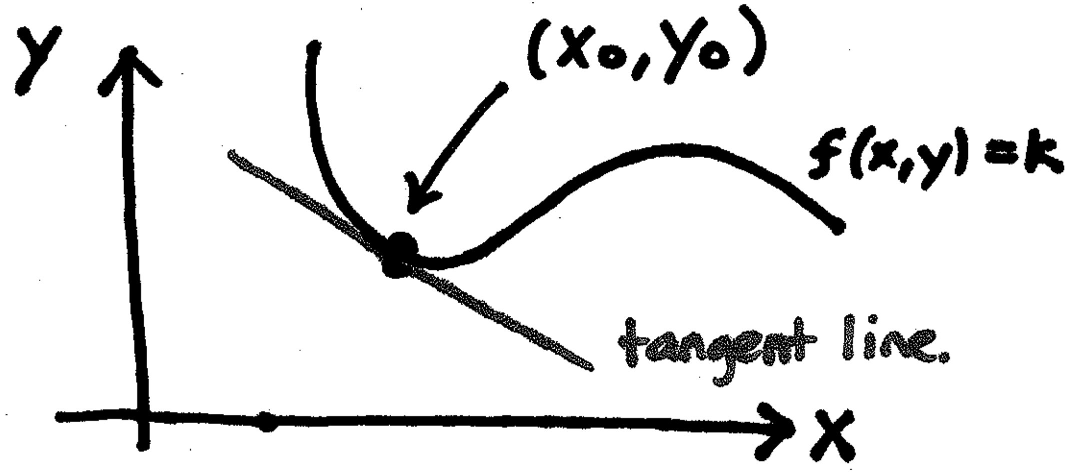
$$x = \frac{16}{\sqrt{3}} \qquad y = \frac{16\sqrt{2}}{\sqrt{3}} \text{ inches.}$$

2. Why does $\nabla f = \lambda \nabla g$ work?

Claim: Contour plot of $z = f(x,y)$. Consider a point on $f(x,y) = k$ with coordinates (x_0, y_0). Then the 2D normal to the contour $f(x,y) = k$ at the point (x_0, y_0) is:

$$\nabla f(x_0, y_0).$$

Proof:



Equation of Contour : $F(x,y) = f(x,y) - k = 0.$

Slope of tangent line $= \dfrac{dy}{dx} = \dfrac{-\dfrac{\partial F}{\partial x}(x_0, y_0)}{\dfrac{\partial F}{\partial y}(x_0, y_0)}$

$$\frac{dy}{dx} = \frac{-\dfrac{\partial f}{\partial x}(x_0, y_0)}{\dfrac{\partial f}{\partial y}(x_0, y_0)}$$

Slope of normal $= \dfrac{-1}{\text{slope of tangent}}$

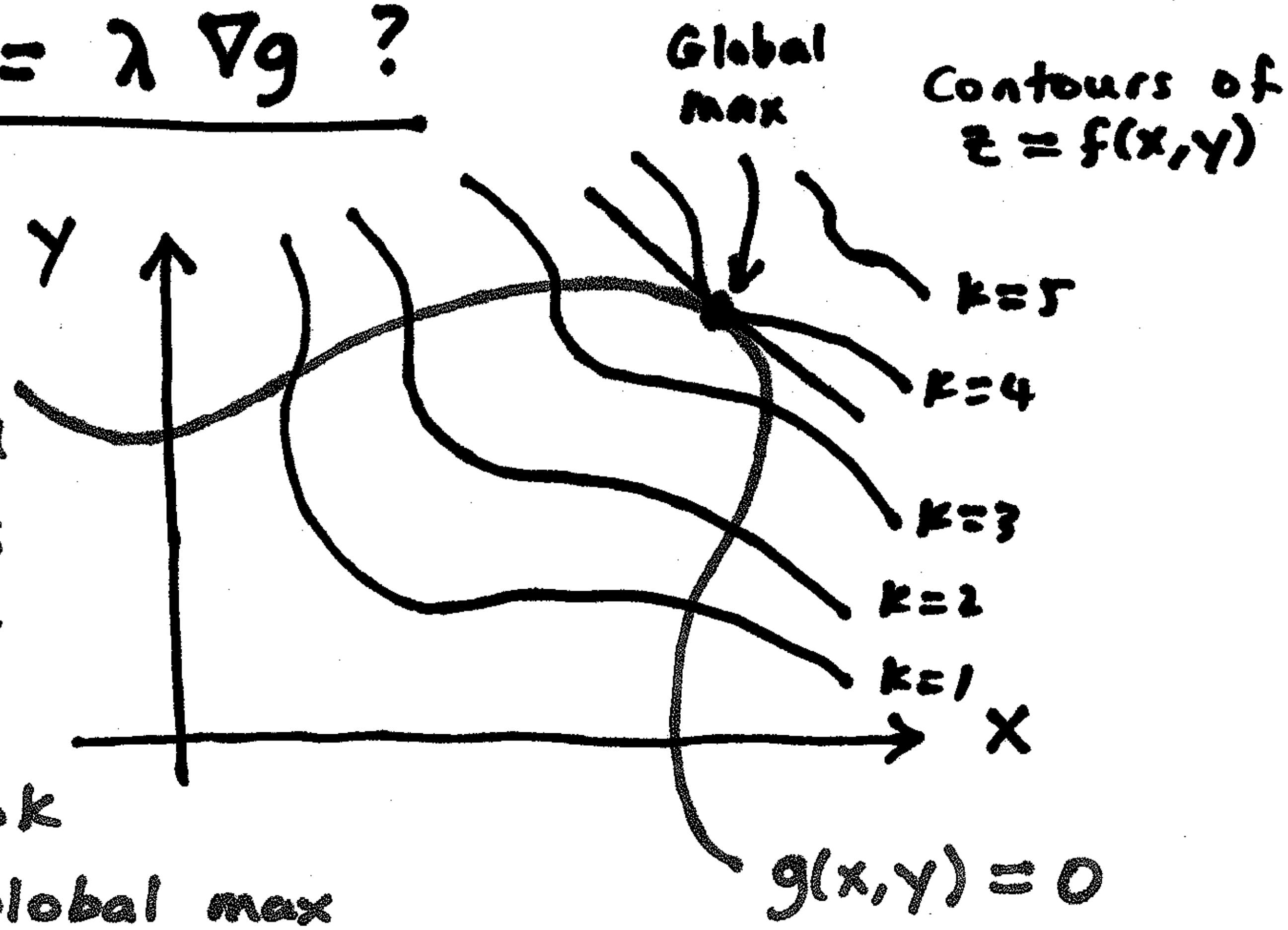
$$= \frac{\dfrac{\partial f}{\partial y}(x_0, y_0)}{\dfrac{\partial f}{\partial x}(x_0, y_0).} = \frac{\text{rise}}{\text{run}}$$

$$\text{Vector in this direction} = \langle \text{run}, \text{rise} \rangle$$

$$= \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle$$

$$= \nabla f(x_0, y_0). \qquad \blacksquare$$

How does this apply to $\nabla f = \lambda \nabla g$?



points where we need to look for global max and min.

At point where global max occurs, $g(x,y) = 0$ and $f(x,y) - 4 = 0$ have the same tangent line.

So, the normal vectors are parallel, and the two normal vectors involved are ∇f, and ∇g.

These are parallel: $\nabla f = \lambda \cdot \nabla g$

number.

<u>Question:</u> What does λ mean?

<u>Answer:</u> Imagine the constraint equation is written:

$$g(x,y) = c.$$

e.g. $$\underbrace{x^2+y^2}_{g(x,y)} = \underbrace{16^2}_{c}$$

If you increase the value in the constraint from c to $c+1$, the value of the global max (and global min) increases by approximately λ.