Outline

1. Directional derivatives.

2. Global max and min.

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Test #2: Friday during lecture.

1. Directional Derivative

- If $z = f(x,y)$ is a function, (x_0, y_0) is a point and a 2D <u>unit</u> vector $\vec{u}$ then:

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \bullet \vec{u}$$

where $\nabla f = \langle \partial f/\partial x, \partial f/\partial y \rangle$.

- The size of $D_{\vec{u}} f(x_0, y_0)$ is maximized if the unit vector:

$$\vec{u} = \frac{1}{|\nabla f|} \cdot \nabla f$$

is used.

- Maximum value of $D_{\vec{u}} f(x_0, y_0)$ is $|\nabla f(x_0, y_0)|$.

Example

A snake is on a plane where temperature is given by:

$$T(x,y) = 20e^{-x^2-3y^2} + 1. \quad (°C)$$

x, y measured in meters.

(a) The snake is at $(2,1)$ and slithers towards $(3,3)$. What is the rate of change in temperature?

(b) What is the greatest rate of change the snake could experience when it slithers away from $(3,3)$, and how big is that rate of change?

Solution

(a) $\nabla T = \langle 20e^{-x^2-3y^2}(-2x), 20e^{-x^2-3y^2}(-6y) \rangle$

$$\nabla T(2,1) \approx \langle -0.0729, -0.1094 \rangle$$

$$\vec{v} = \langle 3,3 \rangle - \langle 2,1 \rangle = \langle 1,2 \rangle$$

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \langle 1/\sqrt{5}, 2/\sqrt{5} \rangle$$

$$D_{\vec{u}} T(2,1) = \langle -0.0729, -0.1094 \rangle \bullet \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$$

$$\approx -0.1142 \ {}^\circ C/m$$

(b) At $(3,3)$, direction for the largest rate of change:

$$\nabla T(3,3) = \langle -2.783 \times 10^{-14}, -8.35 \times 10^{-14} \rangle.$$

Largest rate of change when the snake slithers away from (3,3) is:

$$|\nabla T(3,3)| \approx +8.802 \times 10^{-14} \,^{\circ}C/m.$$

2. Global Max and Min

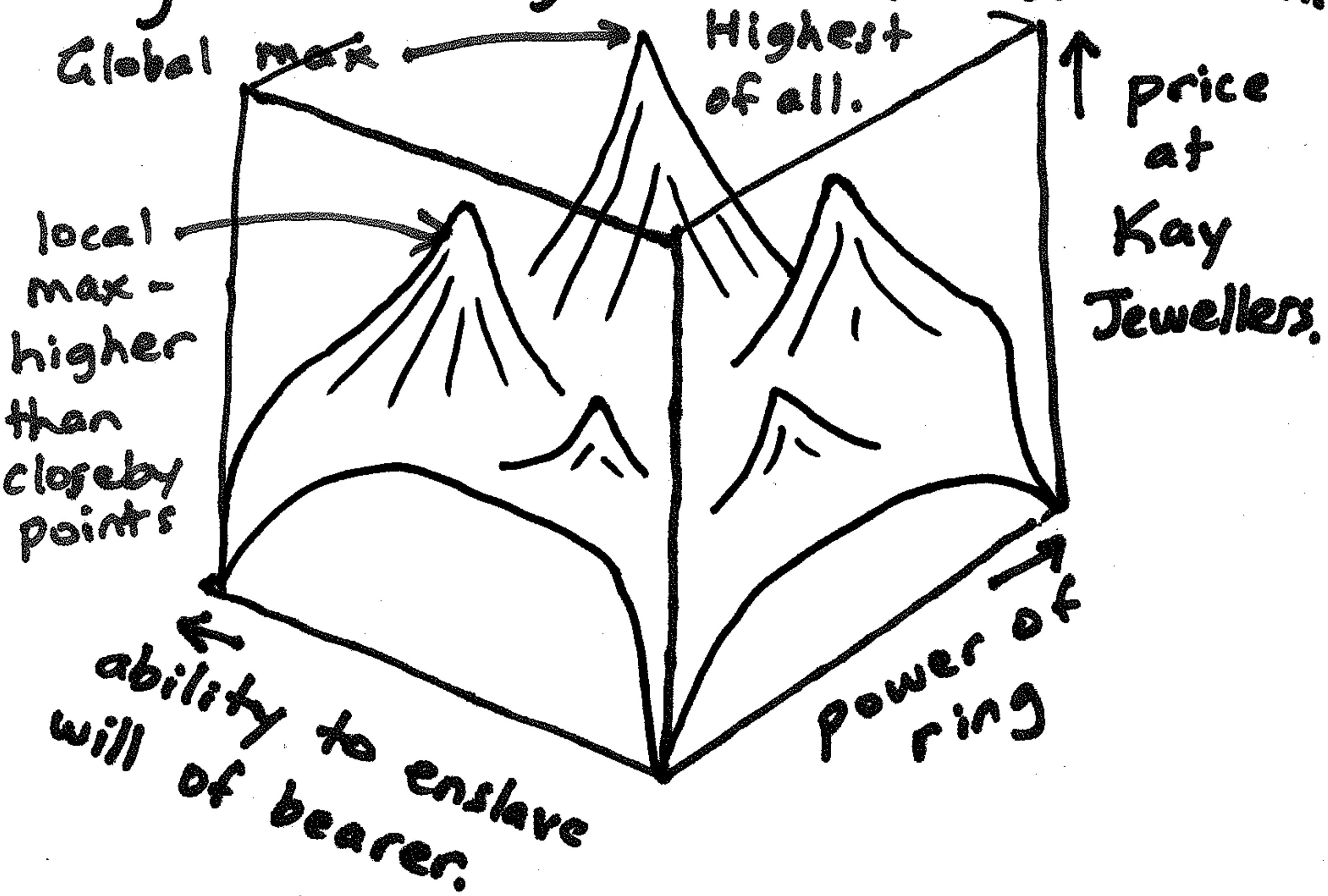
- Global max of $f(x,y)$ over a region R of the x-y plane is the largest z-value that can be obtained by plugging a point $(x,y) \in R$ into $f(x,y)$.

- Global minimum is the lowest z-value that can be obtained.

Local vs. Global Max & Min

- local – higher or lower than points immediately around it.

- global – highest or lowest overall.



Theorem: If C is a closed curve in the xy plane (starts and ends at same point), and R consists all the points on or within C, and $f(x,y)$ is continuous and the domain of $f(x,y)$ includes R, then

$f(a,b)$ = global max or global min of $f(x,y)$ on R

means one of the following must be true:

(I) (a,b) is a point where

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0. \qquad \boxed{\text{OR}}$$

(II) (a,b) is a point where at least one partial derivative

$\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ is not defined. $\boxed{\text{OR}}$

(III) (a,b) is on the curve C.

<u>Example</u>

Find the global max and global min of:

$$f(x,y) = x^2 + y^2 - 2x - y$$

over the triangular region with vertices $(0,0)$, $(2,0)$, $(0,2)$.

<u>Solution</u>

① Find all points where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$.

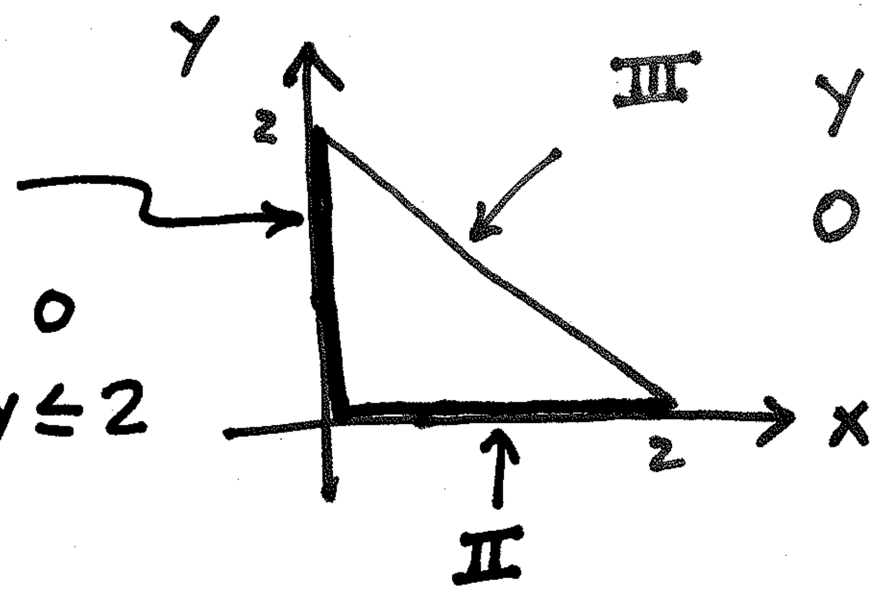
$$\frac{\partial f}{\partial x} = 2x - 2 \qquad \frac{\partial f}{\partial y} = 2y - 1$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{at} \quad (x,y) = (1, \tfrac{1}{2}).$$

② Find all points where $\partial f/\partial x$ or $\partial f/\partial y$ are undefined.

- No such points.

③ Check the boundary.



III $y = -x + 2$

$0 \leq x \leq 2$

I

$x = 0$

$0 \leq y \leq 2$

II

$y = 0 \qquad 0 \leq x \leq 2$

Side I:

$$f(x,y) = y^2 - y = f_I(y).$$

Global max/min: $f_I'(y) = 2y-1=0$

$y = 1/2$

$x = 0$

occurs at one of these points.

End points: $(0,0)$ $(0,2)$