# Outline

1. Classifying critical points.
2. Directional derivatives.
3. Gradient vector.

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Test #2: Friday March 20

# 1. Classifying Max and Mins

## Example

Find the distance between the point $(1, 0, -2)$ and the plane:

$$x + 2y + z = 4.$$

## Solution

Function we want to minimize is:

$$L = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2}$$

Equation of plane: $z = 4 - x - 2y$

$$L(x,y) = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

From previous lesson the critical point is where $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial y}$

both equal zero. This is:

$$x = 11/6 \qquad y = 10/6$$

## Classifying Critical Points

- If we have: $z = f(x,y)$
  we can classify the critical points as:
  - Local mins 
  - Local max 
  - Saddles 

by calculating:

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$ Jacobian Determinant

and evaluate it at the critical point.

## Guide to Interpreting D:

| D evaluated at crit. pt. | $f_{xx}$ evaluated at crit. pt. | Type of crit. pt. |
|---|---|---|
| + | + | local min |
| + | − | local max |
| $\leq 0$ | Doesn't matter | Saddle point. |

## Example

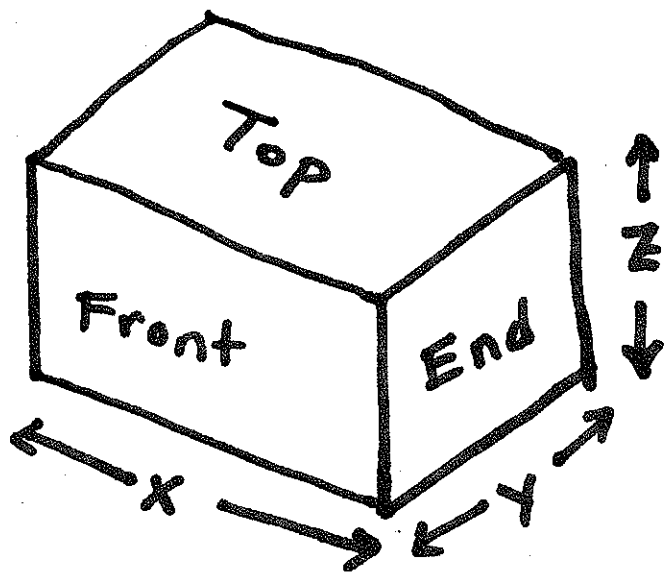
Find the dimensions that minimize the cost of a rectangular box with:

- Volume $= 48 \text{ ft}^3$
- Front & back: \$1 / sq. ft.

- Top & bottom: \$2 /sq. ft.
- Ends: \$3 / sq. ft.

## Solution

$C =$ cost of box in \$.




$$C = (2)(1)xz + (2)(2)xy + (2)(3)yz$$
$$= 2xz + 4xy + 6yz.$$

Volume constraint: $V = x \cdot y \cdot z = 48$

$$z = \frac{48}{x \cdot y}$$

Use this to eliminate $z$ from cost equation.

$$C(x,y) = \frac{96}{y} + \frac{288}{x} + 4xy$$

Now find critical points of $C(x,y)$.

$$\frac{\partial C}{\partial x} = \frac{-288}{x^2} + 4y = 0$$

$$\frac{\partial C}{\partial y} = \frac{-96}{y^2} + 4x = 0$$

Solve these to get one critical point at: $x = 6$ $y = 2$.

Next, classify this critical point.

$$D = C_{xx} \cdot C_{yy} - (C_{xy})^2$$

$$C_{xx} = \frac{576}{x^3} \qquad C_{yy} = \frac{192}{y^3}$$

$$C_{xy} = 4$$

$$D = \left(\frac{576}{x^3}\right)\left(\frac{192}{y^3}\right) - 16$$

Plug in $x=6$ and $y=2$

$D = 48 > 0$ so critical point is a local max or min.

To find out which, plug $x=6$ into $C_{xx}$

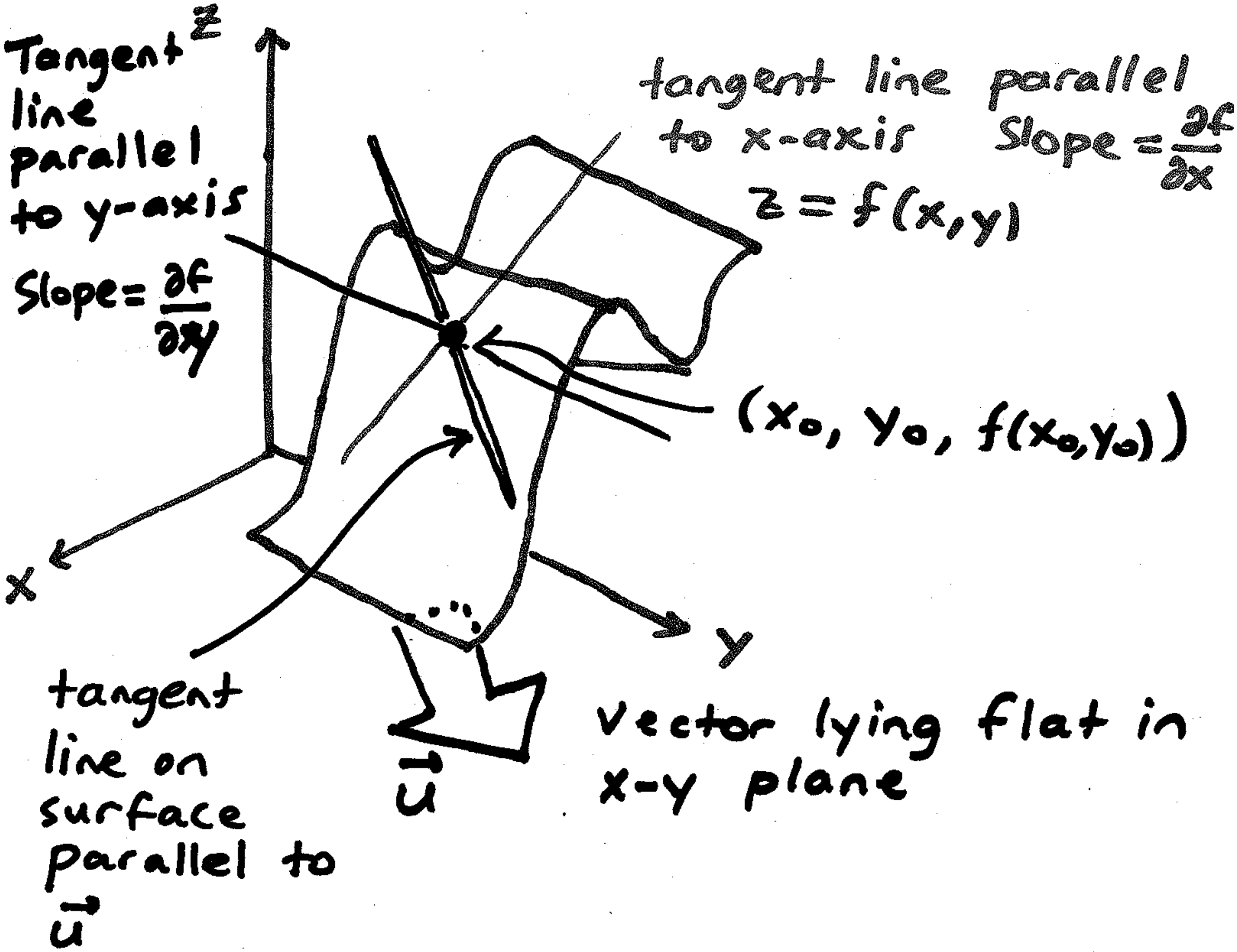
$$C_{xx}(6) = \frac{576}{(6^3)} > 0.$$

So $(x,y) = (6,2)$ is a local min.

$C(6,2) = 144.$ Most economical box costs \$144.

# 2. Directional Derivatives

- Imagine a surface $z = f(x,y)$.




Slope = directional derivative of $f(x,y)$ in direction $\vec{u}$.

- Two formulas for the directional derivative of $f(x,y)$ at the point $(x_0,y_0)$ in direction of the <u>unit</u> vector $\vec{u} = \langle a,b \rangle$ :

<u>Formula 1:</u>

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

<u>Formula 2:</u>

Gradient vector of $f(x,y)$:

$$\nabla f = \langle \partial f/\partial x , \partial f/\partial y \rangle$$

$$D_{\vec{u}} f(x_0,y_0) = \underbrace{\nabla f(x_0,y_0)}_{\substack{\nabla f \text{ evaluated} \\ \text{at } (x_0,y_0).}} \bullet \vec{u}$$