Outline

1. Chain rule for partial derivatives.
2. Implicit differentiation.
3. Critical points.

- HW due Tuesday
- No class Thursday, Friday.

1. Chain Rule for Partial Derivatives

(a) Case 1: $x = x(t)$ $y = y(t)$

- $z = f(x,y)$ is a function of x,y
- $x = x(t)$, $y = y(t)$ are functions of t.
- Goal: Find dz/dt.

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

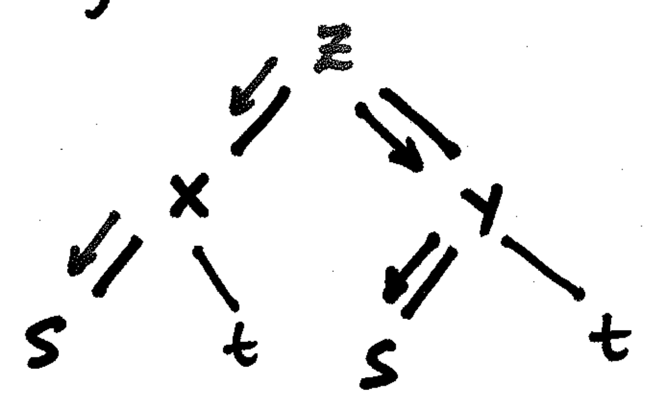
(b) Case 2: $x = x(s,t)$ $y = y(s,t)$

- $z = f(x,y)$ is a function of x,y.
- $x = x(s,t)$ $y = y(s,t)$ are variables of s and t.

- **Goal:** Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} \end{aligned}}$$

Tree diagram:



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

(c) General Case

- $z = f(x_1, x_2, \ldots, x_n)$.

- Each $x_j = x_j(t_1, t_2, \ldots, t_m)$

- **Goal:** $\dfrac{\partial z}{\partial t_i}$.

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_i}$$

$$= \sum_{k=1}^{n} \frac{\partial z}{\partial x_k}\cdot\frac{\partial x_k}{\partial t_i}$$

Example

$$u = \sqrt{r^2+s^2}$$

$$r = y + x\cdot\cos(t)$$

$$s = x + y\cdot\sin(t)$$

Find:

$\partial u/\partial x$

$\partial u/\partial y$

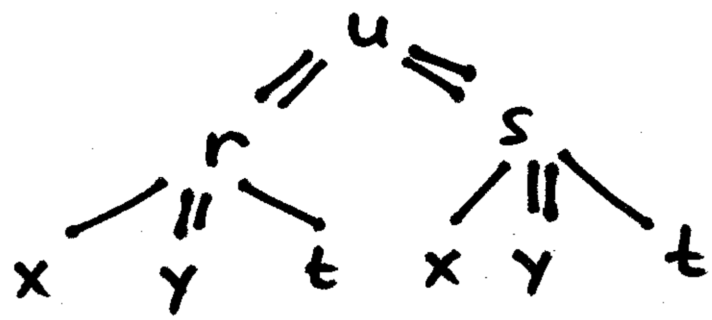
$\partial u/\partial t$.

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$= \frac{r}{\sqrt{r^2+s^2}} \cdot \cos(t) + \frac{s}{\sqrt{r^2+s^2}} \cdot (1)$$

Tree diagram:



$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$= \frac{r}{\sqrt{r^2+s^2}} \cdot (1) + \frac{s}{\sqrt{r^2+s^2}} \cdot \sin(t)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial t} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial t}$$

$$= \frac{r}{\sqrt{r^2+s^2}} (-x \cdot \sin(t)) + \frac{s}{\sqrt{r^2+s^2}} (y \cdot \cos(t))$$

2. Implicit Differentiation

- If we can rearrange an equation involving x and y,

e.g. $\sqrt{x \cdot y} = 1 + x^2 y$

into the form:

$$F(x,y) = 0$$

e.g. $F(x,y) = \sqrt{xy} - 1 - x^2 y$

then:

$$\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}}$$

Why does this work?

- Assume $y = y(x)$ is a function of x.

$$F(x, y) = 0$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

$dx/dx = 1$ So:

$$\frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial F}{\partial x}$$

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

<u>Example</u>

$$\sqrt{xy} = 1 + x^2 y$$

Find dy/dx.

<u>Solution</u> $F(x,y) = \sqrt{xy} - 1 - x^2 y$

$$\frac{dy}{dx} = \frac{-\left(\frac{1}{2} x^{-1/2} y^{1/2} - 2xy\right)}{\frac{1}{2} y^{-1/2} x^{1/2} - x^2}$$

3. Finding and Classifying Critical Points

- At the critical points of a surface $z = f(x,y)$ we must have:

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$

- Guarantees that the tangent plane is horizontal at the critical point.

Example

$$z = f(x,y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

Locate the x, y coordinates of any critical points.

Solution

Find points where $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial x} = 2(x-1) + 2(6-x-2y)(-1)$$

$$= -14 + 4x + 4y$$

$$\frac{\partial f}{\partial y} = 2y + 2(6-x-2y)(-2)$$

$$= -24 + 4x + 10y$$

$\frac{\partial f}{\partial x} = 0$: $\quad 4x + 4y = 14$

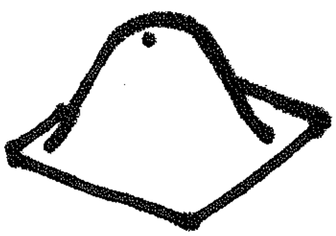
$\frac{\partial f}{\partial y} = 0$ $\quad 4x + 10y = 24$

$x = 11/6 \quad y = 10/6.$

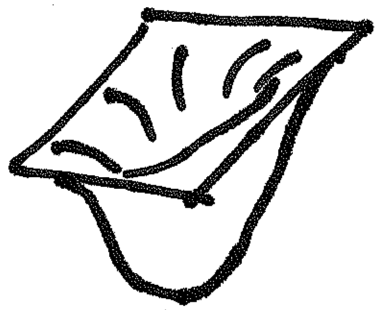
Classifying Critical Points

- There are 3 possibilities for each critical point:

Maximum



Minimum



Saddle