Outline

1. Equation of Tangent Plane.
2. Linear approximation
3. Differentials.
4. Finding and classifying critical points.

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No class Thursday, Friday.

1. Equation for Tangent Plane

- Surface given by $z = f(x,y)$ and a point (x_0, y_0), tangent plane is:

$$\frac{\partial f}{\partial x} \cdot (x - x_0) + \frac{\partial f}{\partial y} \cdot (y - y_0) - (z - z_0) = 0$$

Both evaluated at the point (x_0, y_0).

$z_0 = f(x_0, y_0)$

Example

Find the equation of the tangent plane to:

$$z = f(x,y) = \ln(2x+y)$$

at the point $(x_0, y_0) = (-1, 3)$.

Solution:

Find z_0: $z_0 = \ln(2(-1) + 3)$
$= \ln(1)$
$= 0.$

$$\frac{\partial f}{\partial x} = \frac{1}{2x+y} \cdot (2) \qquad \frac{\partial f}{\partial x}(-1,3) = 2$$

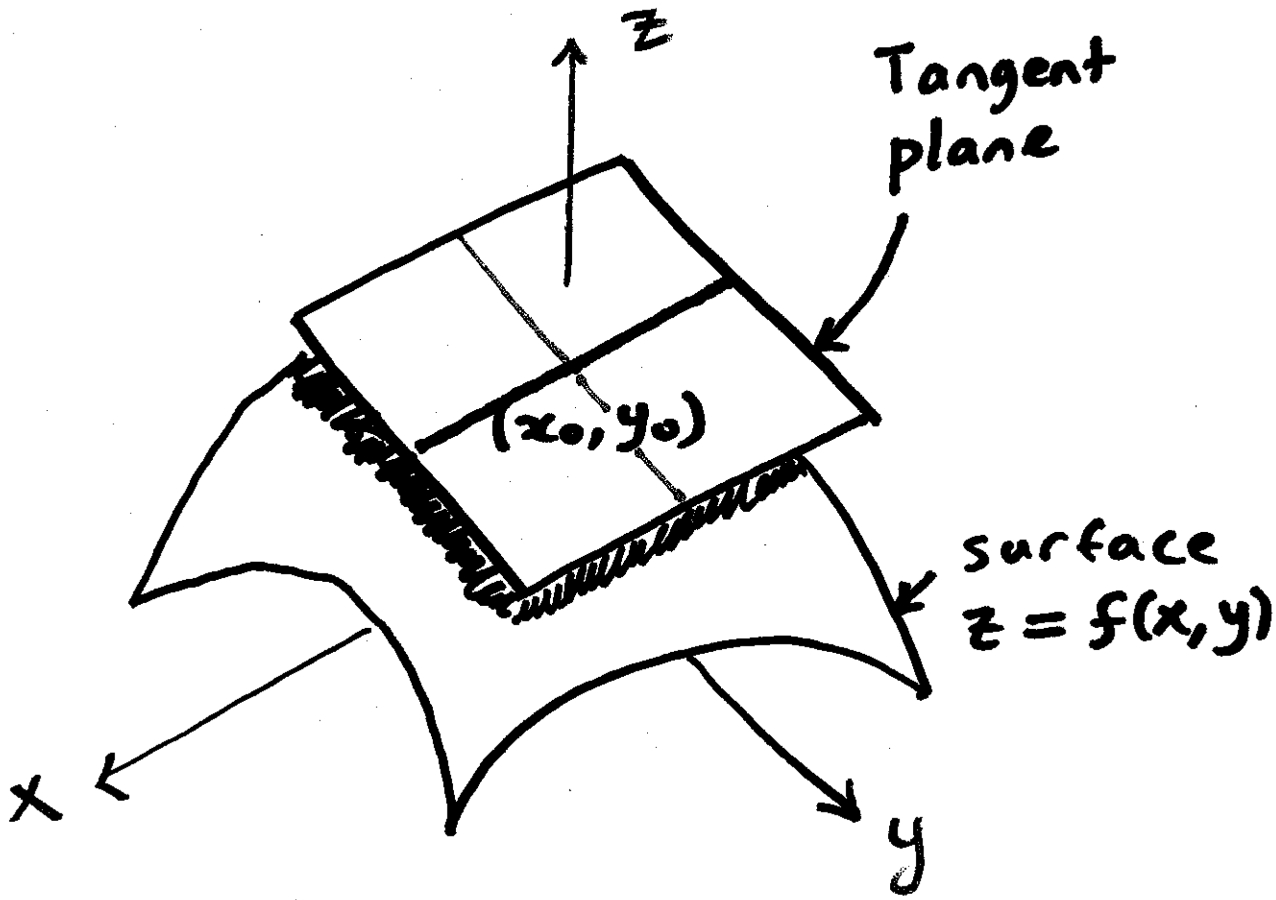
$$\frac{\partial f}{\partial y} = \frac{1}{2x+y} \cdot (1) \qquad \frac{\partial f}{\partial y}(-1,3) = 1.$$

Tangent plane:

$$2(x+1) + 1 \cdot (y-3) - 1 \cdot (z-0) = 0.$$

Why is the normal vector

$\langle \partial f/\partial x , \partial f/\partial y , -1 \rangle$?



The line in the tangent plane passing through (x_0, y_0, z_0) and is parallel to the x-axis:

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\cdot\langle 1, 0, \frac{\partial f}{\partial x}\rangle$$

Similar line but parallel to y-axis:

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\cdot\langle 0, 1, \partial f/\partial y\rangle$$

Normal vector is the cross product of the direction vectors.

$$\langle 0, 1, \partial f/\partial y \rangle \times \langle 1, 0, \partial f/\partial x \rangle$$

$$= \begin{array}{ccccc} i & j & k & i & j \\ 0 & 1 & \partial f/\partial y & 0 & 1 \\ 1 & 0 & \partial f/\partial x & 1 & 0 \end{array}$$

$$= \left\langle \partial f/\partial x , \frac{\partial f}{\partial y} , -1 \right\rangle$$

2. Linear Approximations

- Rearrange tangent plane equation to give linear approximation of $f(x,y)$ at (x_0, y_0) :

$$\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)}$$

Example

Find the approximate value of:

$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}.$$

Solution

Find the linear approximation of:

$$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

at : $(x_0, y_0, z_0) = (3, 2, 6)$.

Then plug $x = 3.02$, $y = 1.97$ and $z = 5.99$ into linear approx.

Linear approx:

$$W = f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0)$$

$$f(x_0, y_0, z_0) = \sqrt{3^2 + 2^2 + 6^2} = 7.$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \quad \text{at } (3,2,6) \quad \frac{\partial f}{\partial x} = \frac{3}{7}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \quad \text{at } (3,2,6) \quad \frac{\partial f}{\partial y} = \frac{2}{7}$$

$$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} \quad \text{at } (3,2,6) \quad \frac{\partial f}{\partial z} = \frac{6}{7}.$$

Linear approx:

$$w = 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6).$$

Estimate of $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}$

$$w = 7 + \frac{3}{7}(3.02-3) + \frac{2}{7}(1.97-2) + \frac{6}{7}(5.99-6)$$

$$= 6.99$$

6.99152343914

↑
actual value.

3. Differentials

- If we rearrange the tangent plane equation and write:

$$x - x_0 = dx \text{ or } \Delta x$$
$$y - y_0 = dy \text{ or } \Delta y$$
$$z - z_0 = dz \text{ or } \Delta z$$

we get:

$$\boxed{dz = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy}$$

Example

What is the uncertainty in the volume of a can (cylindrical can) of height 10 cm, radius 2 cm, when the uncertainties in height and radius are:

$$dh = 0.1 \text{ cm}$$

$$dr = 0.05 \text{ cm} \quad ?$$

Solution

$$V = \pi r^2 \cdot h$$

If $r = 2$ and $h = 10$

$$V = 40\pi \text{ cm}^3$$

$$dV = 2\pi r \cdot h \cdot dr + \pi r^2 \cdot dh$$

$$dV = 2\pi(2)(10)(0.05) + \pi(2^2)(0.1)$$

$$= 2\pi + 0.4\pi$$

Volume $= 40\pi \pm 2.4\pi \text{ cm}^3$

Uncertainty in volume $= \pm 2.4\pi \text{ cm}^3$.