Outline

1. Intersections of surfaces.
2. Partial derivatives.
3. Tangent plane.

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Quiz Thursday.

No class 3/5/09 & 3/6/09.

1. Intersections of Surfaces

- Application of 3D vector functions.
- Process:

① Make z subject in both surface equations.

② Equate the two equations.

③ Solve for x or y.
(Say we've solved for y.)

④ Set $x = t$.
Substitute $x=t$ into y formula.
Substitue expressions for x and y into one of the surface equations.

⑤ Put x, y, z together in a vector.

Example

Write a vector function for the curve of intersection of:

- Cone: $z^2 = x^2 + y^2$, $z \geqslant 0$
- Plane: $z = 1 + y$.

Solution

① Cone: $z = \sqrt{x^2+y^2}$

② $1 + y = \sqrt{x^2+y^2}$

③ $(1+y)^2 = x^2+y^2$

$$y = \frac{1}{2}(x^2-1)$$

④ $x = t$

$$y = \frac{1}{2}(t^2-1)$$

⑤ $z = 1+y = 1+\frac{1}{2}(t^2-1) = \frac{1}{2}(t^2+1)$

(6) $\vec{r}(t) = \langle t, \frac{1}{2}(t^2-1), \frac{1}{2}(t^2+1) \rangle$

2. Partial Derivatives

- Derivatives of functions like $f(x,y)$ where we look at the rate of change of $f(x,y)$ with respect to either x or y, but keep the other variable fixed.

- Limit definitions:

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

- All the usual derivative rules work for partial derivatives.

Example

$$f(x,y) = e^{-\frac{1}{3}x^3 + x - y^2}$$

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

- Calculate $\frac{\partial f}{\partial x}$, pretend y is constant.

$$\frac{\partial f}{\partial x} = e^{\left(-\frac{1}{3}x^3 + x - y^2\right)} \cdot \left(-x^2 + 1 + 0\right)$$

- Calculate $\frac{\partial f}{\partial y}$, pretend x is constant.

$$\frac{\partial f}{\partial y} = e^{\left(\frac{-1}{3}x^3 + x - y^2\right)} \cdot (0 + 0 - 2y)$$

Notation

$\frac{\partial^2 f}{\partial x \, \partial y}$ means differentiate $f(x,y)$ with respect to x, then differentiate the result with respect to y.

Clairaut's Theorem

So long as the 2nd derivatives ($f_{xx} = \partial^2 f/\partial x \partial x$, $f_{yy} = \partial^2 f/\partial y \partial y$, f_{xy} , f_{yx}) are continuous :

$$f_{xy} = f_{yx}.$$

3. Tangent Plane

- Surface: $z = f(x,y)$, instead of a tangent line at each point, we have a tangent plane at each point.

Normal Vector of a Tangent Plane

Surface: $z = f(x,y)$.

Point: x-coord $= x_0$
y-coord $= y_0$
z-coord $= f(x_0,y_0) = z_0$.

Normal: $\left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1 \right\rangle$

Example

Find the equation of the tangent plane to the surface:

$$z = e^{\left(\frac{-1}{3}x^3 + x - y^2\right)}$$

where $x_0 = 1.5$ and $y_0 = 0$.

Solution

$$z_0 = e^{\left(\frac{-1}{3}(1.5)^3 + 1.5 - 0^2\right)} \doteq 1.455$$

$$\approx$$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = (-x^2+1) \cdot e^{\left(\frac{-1}{3}x^3 + x - y^2\right)}$$

$$\frac{\partial z}{\partial x}(1.5, 0) \approx -1.818$$

$$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = -2y \cdot e^{\left(\frac{-1}{3}x^3 + x - y^2\right)}$$

$$\frac{\partial z}{\partial y}(1.5, 0) = 0$$

Normal = $\langle -1.818,\ 0,\ -1 \rangle$

Equation of plane:

$$-1.818\,(x - 1.5) + 0\cdot(y-0) - 1\,(z - 1.455) = 0$$