Outline

1. Finish 2D limits.
2. Vector functions /curves.
3. Tangent vectors.
4. Intersections of surfaces.

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Do-over: Tuesday
8 - 9 pm
9 - 10 pm
2210 DH.

1. Limits in 2D

Definition: $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$

means:

$\boxed{\forall\ \varepsilon > 0 \quad \exists\ \delta > 0}$ such that

$\sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y)-L| < \varepsilon$.

What you do to show a limit exists:

① $|f(x,y) - L|$ write this down.

② Show that for some number K,

$$|f(x,y)-L| \leq K \cdot \sqrt{(x-a)^2+(y-b)^2}$$

③ Given $\varepsilon > 0$, set $\delta = \varepsilon / K$.

Good trick: $|x \cdot y| \leq \frac{1}{2}(x^2 + y^2)$.

Example

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = 0$.

Solution

① $$|f(x,y) - L| = \left| \frac{3xy^2}{x^2+y^2} - 0 \right|$$

$$= \frac{3 \cdot |x| \cdot y^2}{x^2 + y^2}$$

Next Good Trick:

$$\boxed{\begin{array}{l} x^2 \leq x^2 + y^2 \\ y^2 \leq x^2 + y^2 \end{array}}$$

② Observe:

$$\frac{y^2}{x^2+y^2} \le \frac{x^2+y^2}{x^2+y^2} = 1.$$

$$3 \cdot |x| = 3 \cdot \sqrt{x^2} \le 3 \cdot \sqrt{x^2+y^2}$$

$$\frac{3 \cdot |x| \cdot y^2}{x^2+y^2} \le 1 \cdot 3 \cdot \sqrt{x^2+y^2}$$

So $K = 3$.

③ If $\varepsilon > 0$ is given, use $\delta = \varepsilon/3$.

<u>Example</u>

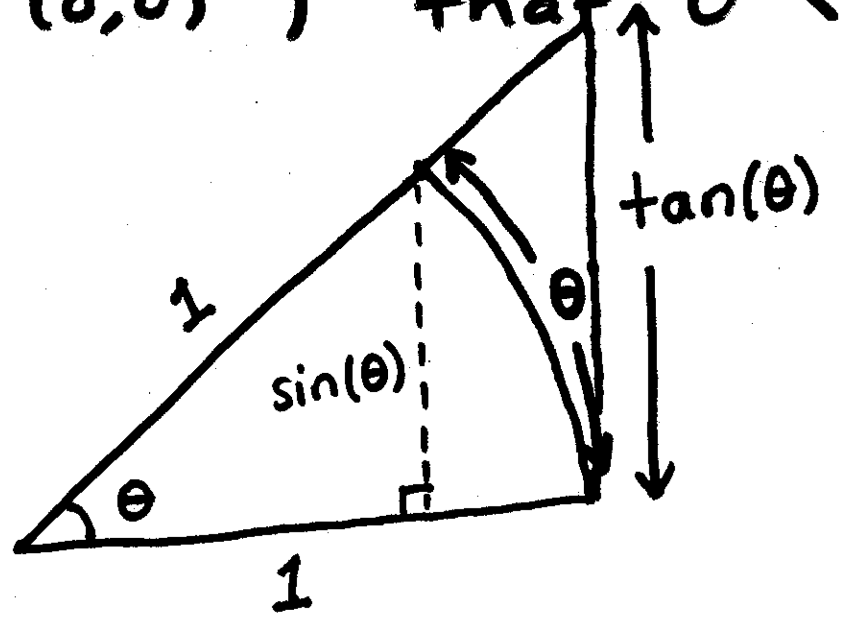
Show that: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$

Solution:

Good Trick #3: Set $\theta = x^2 + y^2$.

Then: $$\frac{\sin(x^2+y^2)}{x^2+y^2} = \frac{\sin(\theta)}{\theta}.$$

Assume (reasonable since $(x,y) \to (0,0)$) that $\theta < \pi/2$.



$$\sin(\theta) < \theta$$

$$\boxed{\frac{\sin(\theta)}{\theta} < 1.}$$

$$\theta < \tan(\theta)$$

$$\theta < \frac{\sin(\theta)}{\cos(\theta)}$$

$$\boxed{\cos(\theta) < \frac{\sin(\theta)}{\theta}}$$

Last Good Trick: If for all (x,y) in a neighborhood of (a,b) we have:

$$h(x,y) \leq f(x,y) \leq g(x,y)$$

and

$$\lim_{(x,y)\to(a,b)} h(x,y) = \lim_{(x,y)\to(a,b)} g(x,y) = L$$

then:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Take on faith:

$$\lim_{(x,y)\to(0,0)} 1 = 1$$

$$\lim_{(x,y)\to(0,0)} \cos(x^2+y^2) = 1$$

We have that when (x,y) is close to $(0,0)$ so that $\theta = x^2+y^2 < \pi/2$:

$$\cos(x^2+y^2) \leq \frac{\sin(x^2+y^2)}{x^2+y^2} \leq 1.$$

By the 2D squeezing lemma:

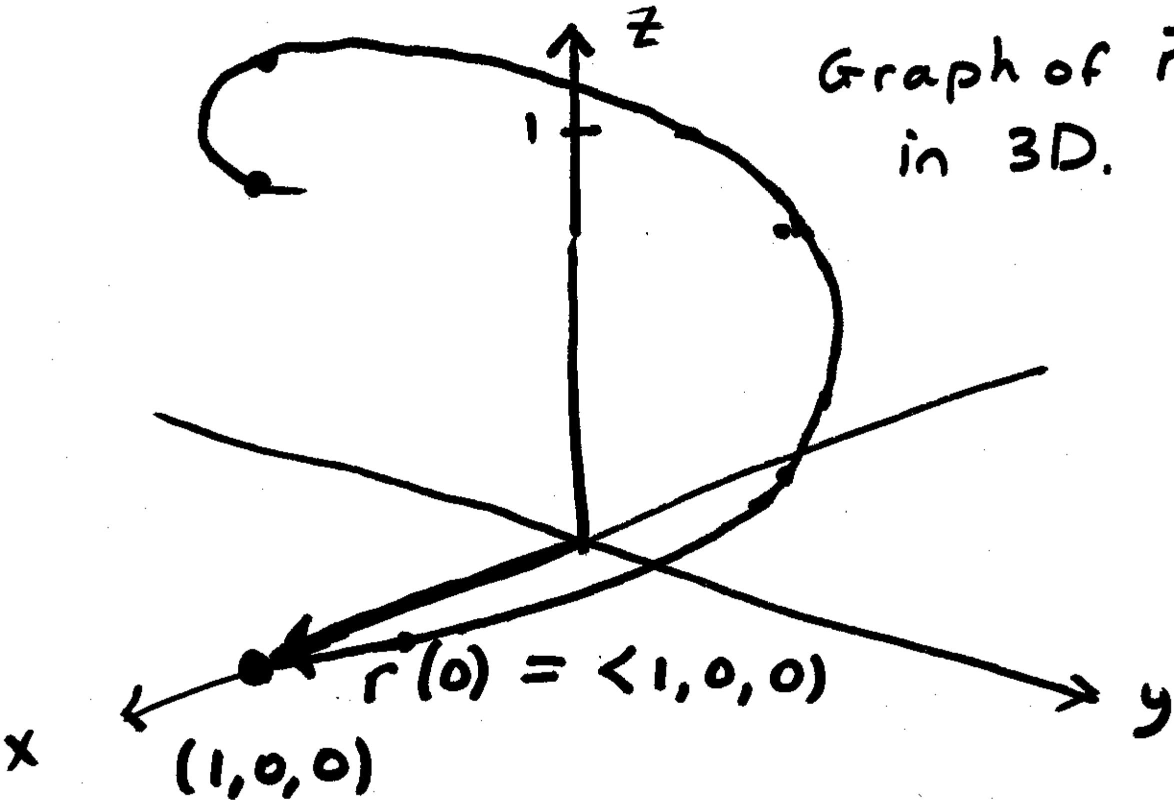
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$$

2. Vector Functions and Curves

- A vector function is a function that gives a vector as its output.

e.g. $\vec{r}(t) = \langle \cos(2\pi t), \sin(2\pi t), t \rangle$

x, y, z coordinates of a 3D curve in space.



Graph of $\vec{r}(t)$ in 3D.

3. Tangent Vectors to Curves

- When you take the derivative of a vector function $\vec{r}(t)$, differentiate each component.

e.g. $\vec{r}(t) = \langle t \cdot \cos(2\pi t), t \cdot \sin(2\pi t), t \rangle$

$$\vec{r}\,'(t) = \langle \cos(2\pi t) - 2\pi t \sin(2\pi t), \\ \sin(2\pi t) + 2\pi t \cdot \cos(2\pi t), \\ 1 \rangle$$

- $\vec{r}\,'(t)$ is a vector that is tangent to the curve formed by the graph of $\vec{r}(t)$.

Example

Find an equation for the tangent line to:

$$\vec{r}(t) = \langle t \cdot \cos(2\pi t),\ t \cdot \sin(2\pi t),\ t \rangle$$

when $t = 1$.

Solution:

$$\vec{r}(1) = \langle 1,\ 0,\ 1 \rangle$$

$$\vec{r}\,'(1) = \langle 1,\ 2\pi,\ 1 \rangle$$

Vector equation for tangent line:

$$\langle x, y, z \rangle = \langle 1, 0, 1 \rangle + t \cdot \langle 1, 2\pi, 1 \rangle$$

- The unit tangent vector, $\vec{T}(t)$,

$$\boxed{\vec{T}(t) = \frac{1}{|\vec{r}\,'(t)|} \cdot \vec{r}\,'(t)}$$