# Outline

1. When contours converge.
2. Limits in 1D.
3. Limits in 2D.

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Do-over: Tuesday
8-9 pm
9–10 pm
2210 Doherty.

# 1. When Contours Converge

## Example

Draw the graph of:

$$z = f(x,y) = \frac{x^2}{x^2+y^2}$$

## Solution

Make a contour plot.

Set $z = k$.

$$k = \frac{x^2}{x^2+y^2}$$

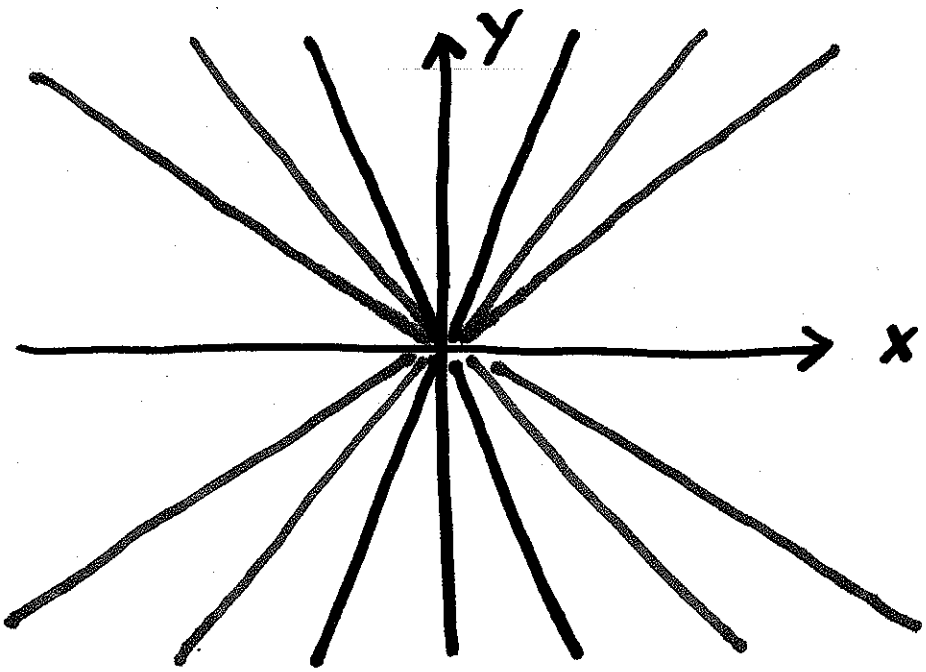
$$k(x^2+y^2) = x^2$$

$$ky^2 = x^2 - kx^2$$

$$y^2 = \left(\frac{1-k}{k}\right)x^2$$

$$y = \pm\sqrt{\frac{1-k}{k}}\ x$$

Contours are straight lines through (0,0).




## 2. Limits in 1D

- In order for a limit to exist:

$$\lim_{x \to a} f(x) = L$$

we have to have:

Left Hand: $\lim_{x \to a^-} f(x) = L.$

Right Hand: $\lim_{x \to a^+} f(x) = L.$

## Formal Definition

$\lim_{x \to a} f(x) = L$ means:

$\forall\ \varepsilon > 0, \ \exists\ \delta > 0$ such that:

$\underset{\text{for every}}{\forall}$ $\underset{\text{there exists}}{\exists}$

$\underbrace{|x-a| < \delta}_{\text{when } x \text{ is close to } a}$ then $\underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is also close to } L.}$.

## 3. Limits in 2D

- In 2D for a limit:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to exist, you have to get the same z-value of L no matter how you approach the point (a,b) in the xy-plane. (Literally an infinite number of ways.).

### How to Show a 2D Limit Does not Exist

Strategy: ① Find two different paths that approach $(a,b)$.

② Show you get a different value of $L$ by following each path.

Example

Show that $$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$ does not exist.

Solution

First path: $y = x$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Second path: y-axis $x = 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2}{0^2+y^2} = 0.$$

Two different values along two different paths, so:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \text{ does not exist.}$$

## How to Show that a 2D Limit Actually Exists

- The definition of a limit for a function $f(x,y)$ is:

$$\forall \varepsilon > 0, \quad \exists \delta > 0$$

such that if:

$$\sqrt{(x-a)^2+(y-b)^2} < \delta \quad \text{then:}$$

$$|f(x,y) - L| < \varepsilon.$$

[redacted] and it means:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

What you actually have to do:

1. Start with $|f(x,y) - L|$.

2. Demonstrate that for some number $k$:

$$|f(x,y) - L| \leqslant k \cdot \sqrt{(x-a)^2+(y-b)^2}.$$

3. When $\varepsilon > 0$ is given, $\delta = \varepsilon / k$.

## Example

Show that:

$$\lim_{(x,y)\to(0,0)} \frac{x \cdot y}{\sqrt{x^2+y^2}} = 0.$$

## Solution

① $$|f(x,y)-L| = \left|\frac{x \cdot y}{\sqrt{x^2+y^2}} - 0\right|$$

$$= \frac{|x \cdot y|}{\sqrt{x^2+y^2}}$$

② Claim: $|x \cdot y| \leq \frac{1}{2}(x^2+y^2)$

Proof: $0 \leq (x-y)^2$

$$0 \leq x^2 - 2xy + y^2$$

$$2xy \leq x^2 + y^2$$

$$xy \leq \frac{1}{2}(x^2+y^2).$$

$$\frac{|x \cdot y|}{\sqrt{x^2+y^2}} \leq \frac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}$$

$\boxed{k = 1/2}$ $\sqrt{(x-0)^2+(y-0)^2}$

③ If $\varepsilon > 0$ is given, set $\delta = 2\varepsilon$.