Outline

1. Graphing quadric surfaces.
2. Formulas for quadric surfaces.
3. Limits in 1D.
4. Limits in 2D.

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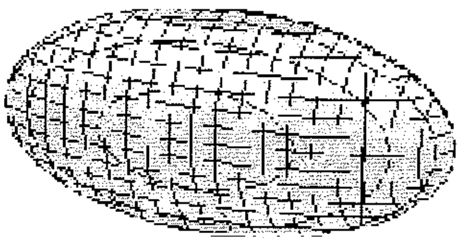
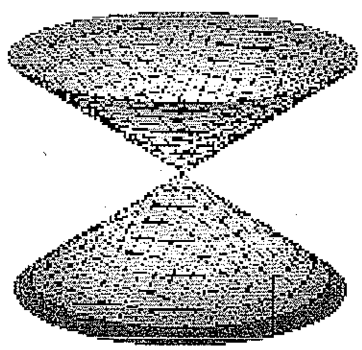
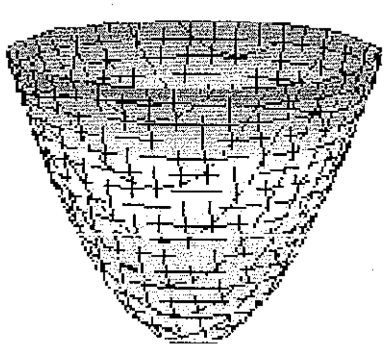
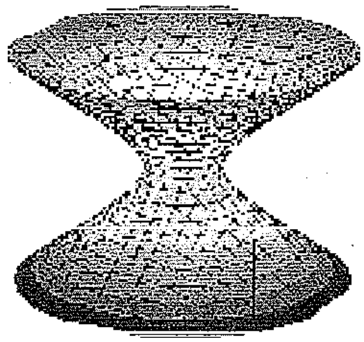
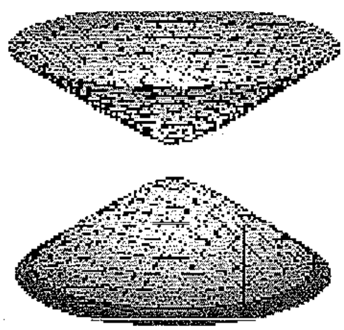
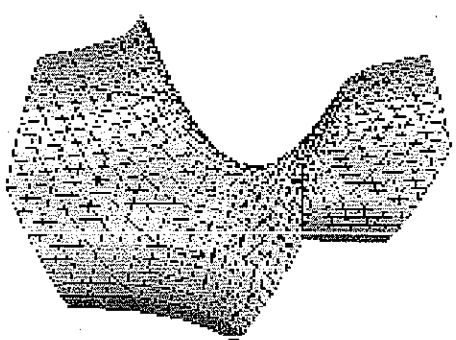
Quiz tomorrow in recitation.

Handout 3: Classifying Quadric Surfaces

The key to classifying conic sections (in polar coordinates) was the eccentricity. It is possible to calculate similar numbers that can be used to classify quadric surfaces but this is much more complicated than eccentricity. We will classify quadric surfaces by the structure of the equation for the surface. The formula for each quadric surface is summarized in the table given below (were we assume that a, b and c are all non-zero constants).

Surface	Graph	Equation	Comments
Ellipsoid		$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$	Sphere is the case when $a = b = c$. The point (x_0, y_0, z_0) is at the very center of the ellipsoid.
Cone		$\frac{(z-z_0)^2}{c^2}=\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}$	The point (x_0, y_0, z_0) is the point where the two parts of the cone meet. If either x or y appears as the subject of the equation, then the cone opens along that axis instead.
Elliptic paraboloid		$\frac{z-z_0}{c}=\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}$	One of the variables will be raised to the first power. This gives the axis that the paraboloid opens along. The case $c > 0$ is illustrated here. The point (x_0, y_0, z_0) is the lowest point on the paraboloid.

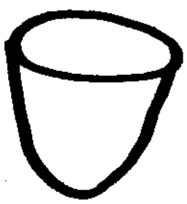
Hyperboloid of one sheet		$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=1$$	Only one of the terms will be subtracted. This gives the axis along which the hyperboloid opens. The point (x_0, y_0, z_0) is at the center of the thinnest part of the surface
Hyperboloid of two sheets		$$-\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$	Only one of the terms will be added. This gives the axis along which the hyperboloid opens. The point (x_0, y_0, z_0) is at the halfway point between the sheets.
Hyperbolic paraboloid		$$\frac{z-z_0}{c}=\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}$$	This surface has a saddle shape like a Pringles potato chip. It is also possible for x to be subtracted and y added. The case $c<0$ is illustrated here.

1. Graphing Quadric Surfaces

- Two families of Quadric Surfaces:

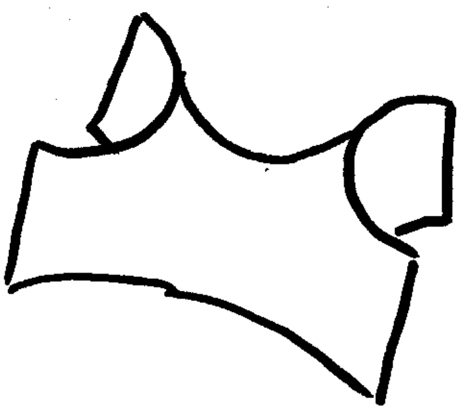
$\underline{z}$ $\underline{z^2}$

Elliptic Paraboloid



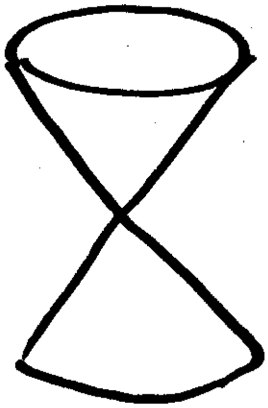
$$\frac{z-z_0}{c} = \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

Hyperbolic Paraboloid



$$\frac{z-z_0}{c} = \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2}$$

z^2

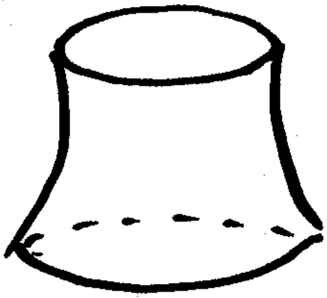


Cone

$$\frac{(z-z_0)^2}{c^2} = \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

Ellipsoid

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$



Hyperboloid of One Sheet

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} - \frac{(z-z_0)^2}{c^2} = 1$$

Hyperboloid of Two Sheets

$$-\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

Drawing Graphs of Quadric Surfaces

- Good things to identify:
  - □ The type of surface.
  - □ Intercepts (of x, y, z axes).
  - □ Hyperboloid (1 or 2 sheets), or paraboloid: Axis of symmetry.
  - □ Open upwards or downwards?

Example

Sketch the set of solutions to:

$$f(x, y, z) = -36$$

where: $f(x, y, z) = -4x^2 - 9y^2 - 36z^2$.

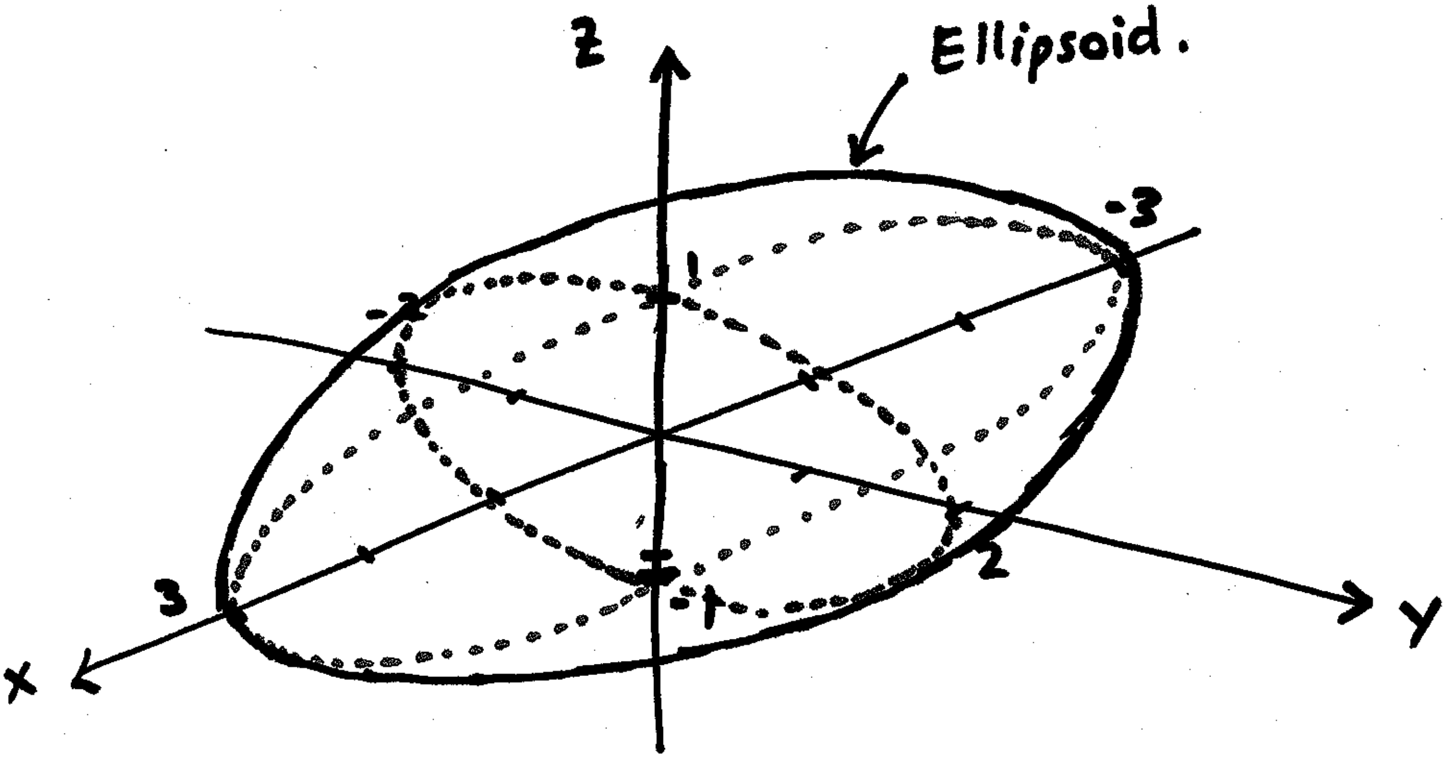
Solution

$$-4x^2 - 9y^2 - 36z^2 = -36$$

$$4x^2 + 9y^2 + 36z^2 = 36$$

$$\frac{x^2}{(3)^2} + \frac{y^2}{(2)^2} + \frac{z^2}{(1)^2} = 1$$

This is an ellipsoid.



Example

Sketch the graph of:

$$4z^2 - x^2 - y^2 = 1.$$

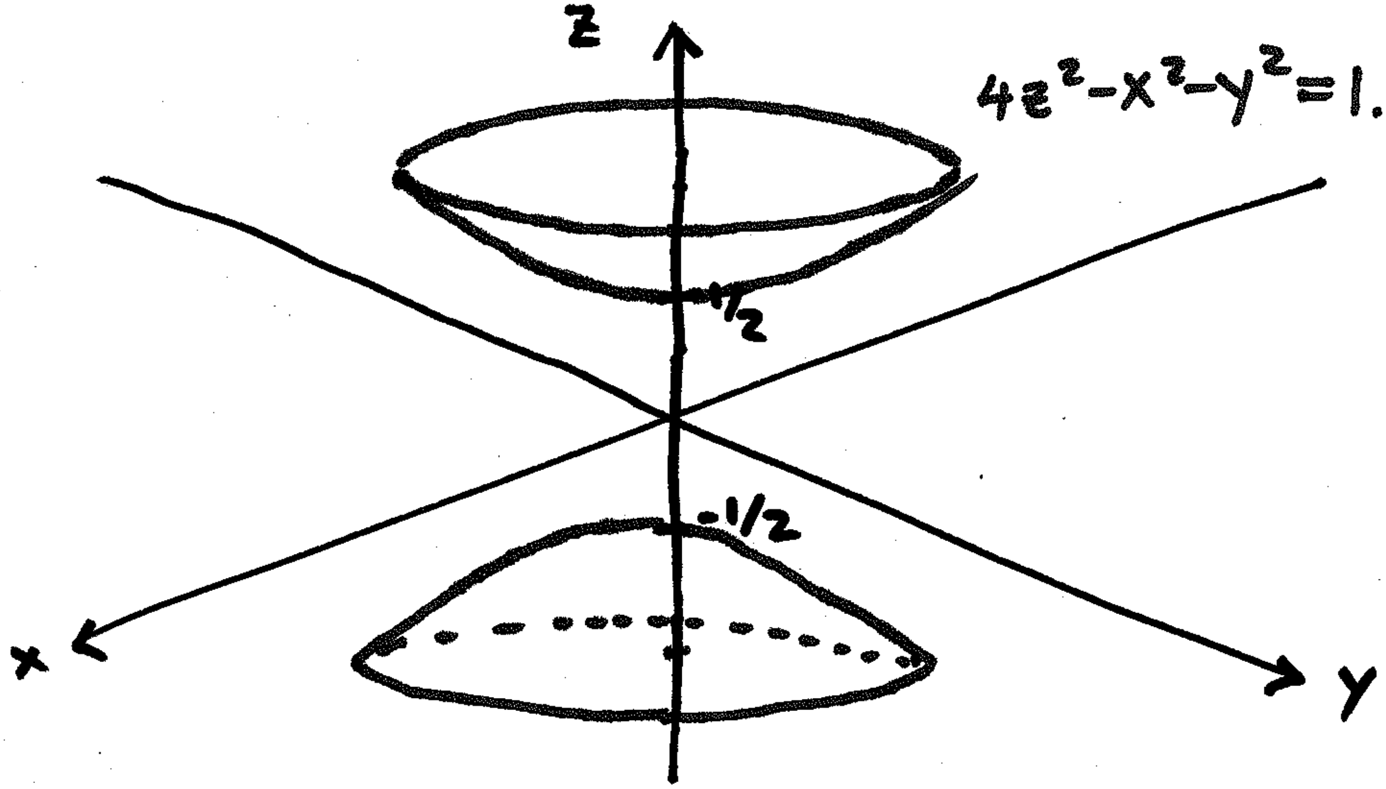
Solution

- Surface is a hyperbola of two sheets.
- Axis of "symmetry" or "rotation" is the z-axis.
- Intercepts: $(0, 0, 1/2)$ $(0, 0, -1/2)$.
- To get the cross-sectional shape of each sheet of the hyperboloid, find the equation of a contour.

$z=1$ $\quad 4 - x^2 - y^2 = 1$

$$\underbrace{x^2 + y^2 = 3}_{\text{circle}},$$

So cross-sections are circular.



Example

Draw the graph of:

$$x^2 + 4z^2 - y = 0$$

one term not squared.

Solution

- Elliptic paraboloid.

$$y = 4z^2 + x^2$$

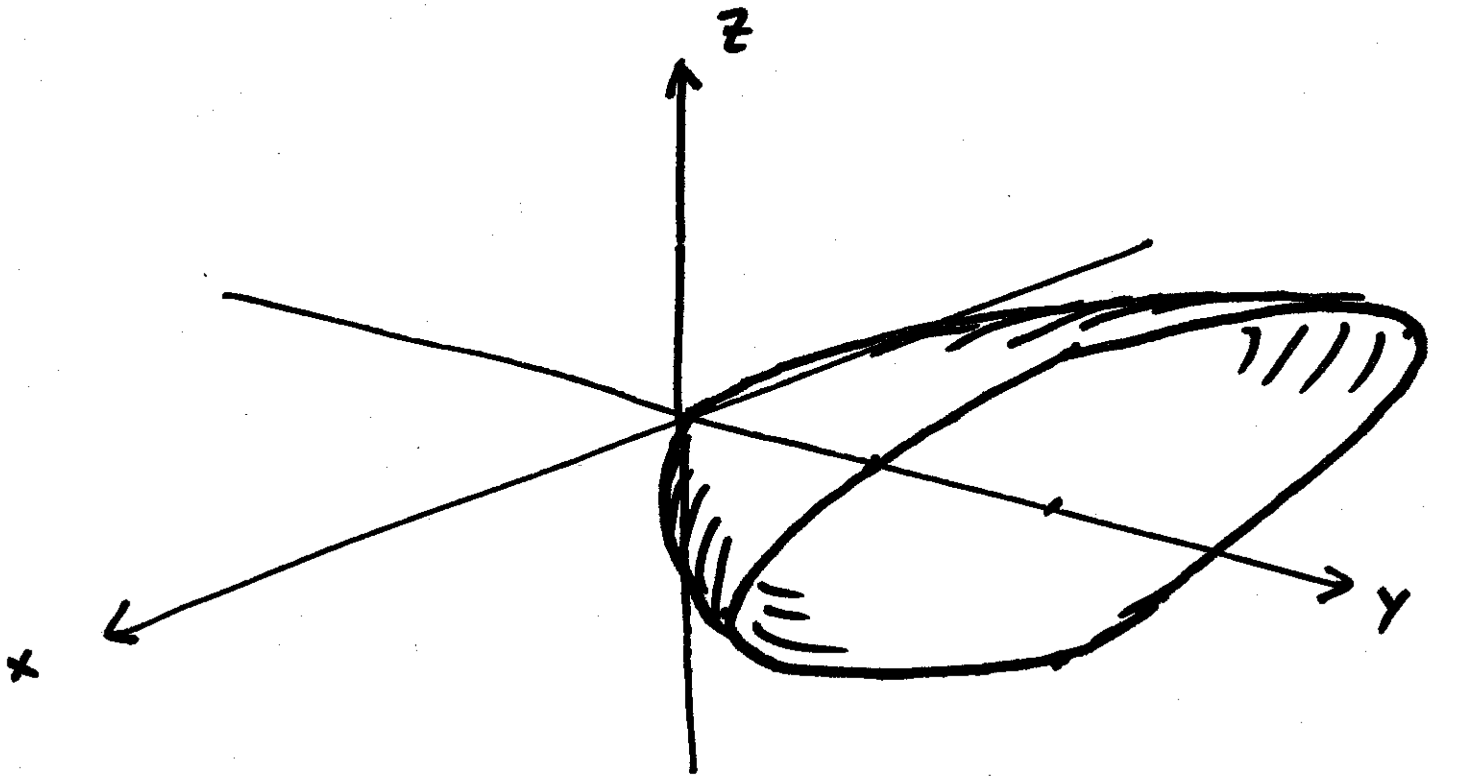
- Axis of "symmetry" is y-axis.
- Opens along positive y-axis.

- To find cross-section.

<u>$y=1$:</u>

$$4z^2 + x^2 = 1$$

$$\frac{x^2}{1^2} + \frac{z^2}{(1/2)^2} = 1 \quad \text{Ellipse.}$$



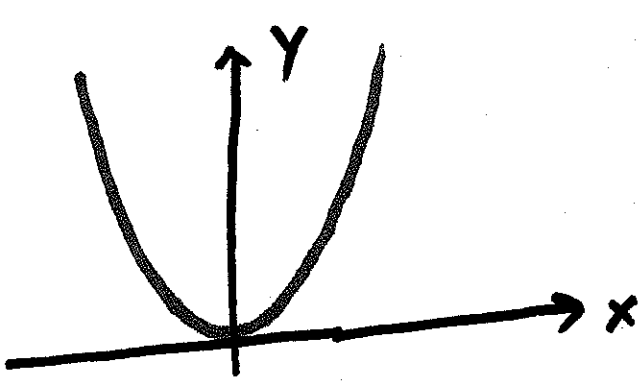
2. Formulas for Quadric Surfaces.

Example

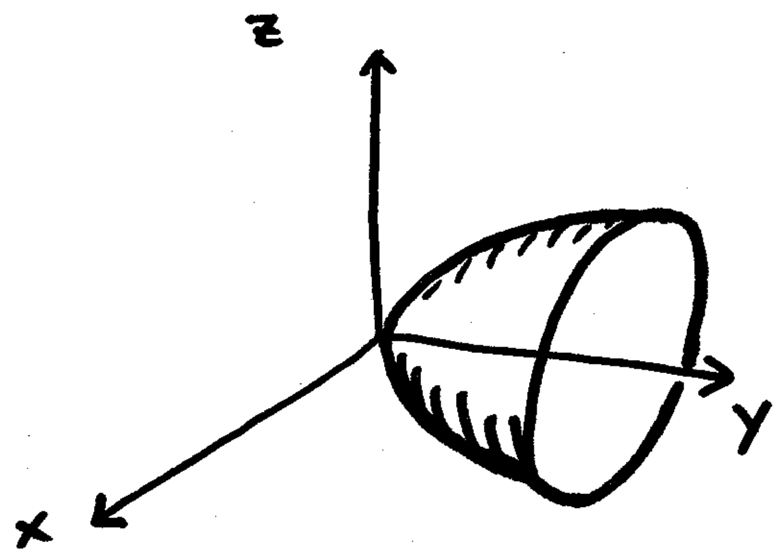
Find a formula for the surface created when $y = x^2$ (in 2D) is rotated around the y-axis (in 3D).

Solution

2D



3D



This is an elliptic paraboloid, so its formula will look like:

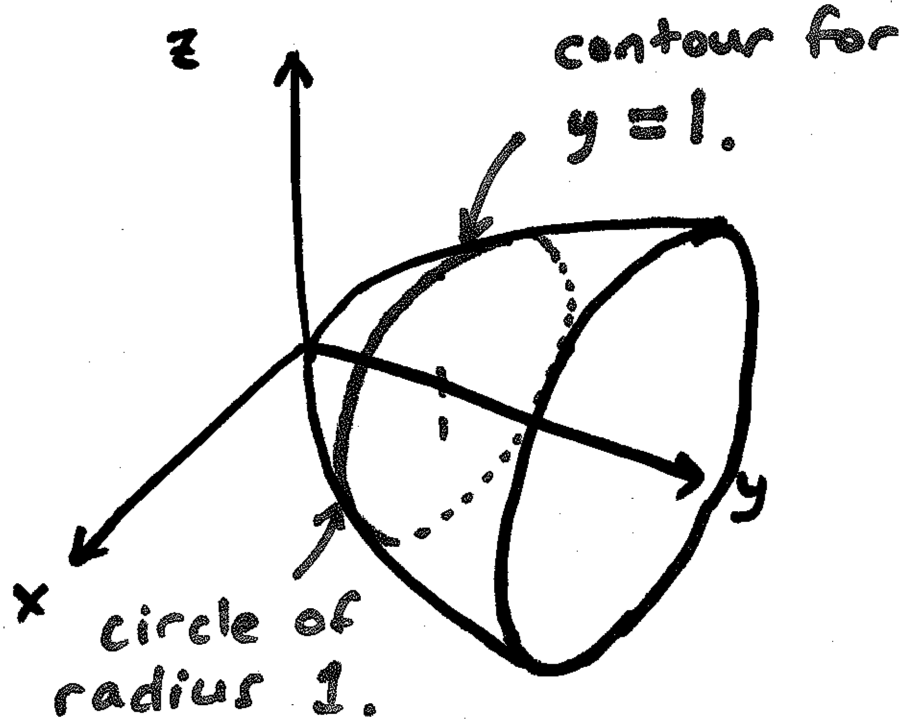
$$\frac{y-y_0}{b} = \frac{(x-x_0)^2}{a^2} + \frac{(z-z_0)^2}{c^2}.$$

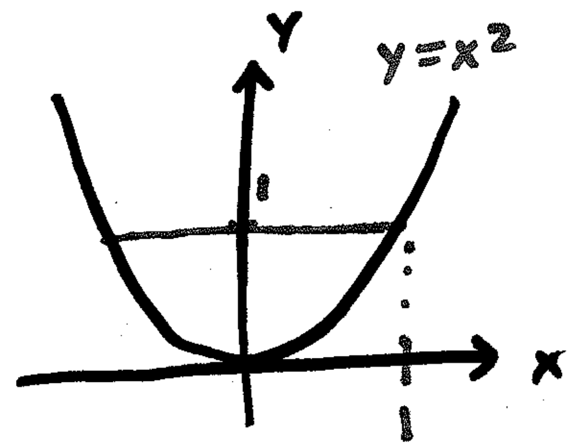
Need to find x_0, y_0, z_0, a, b, c.

(x_0, y_0, z_0) is the "top" of the bell, which occurs at $(0,0,0)$. So:

$$x_0 = 0 \qquad y_0 = 0 \qquad z_0 = 0.$$

$$\frac{y}{b} = \frac{x^2}{a^2} + \frac{z^2}{c^2}$$





Equation of contour is equation of circle with radius 1, i.e.

$$1 = x^2 + z^2$$

This occurred for $y=1$.

$$y = x^2 + z^2$$

So $a = b = c = 1$.