Outline

1. Geometry of the cross product.
2. Parallelpipeds.
3. Lines and planes in 3D.
4. Symmetric equations.

No HW or quiz this week.

Office hours Tuesday: 11am - 1pm.

1. Geometry of the Cross Product

(a) Parallel and Orthogonal

(i) Orthogonal

$\vec{a}$ and $\vec{b}$ are orthogonal if: $\vec{a} \bullet \vec{b} = 0.$

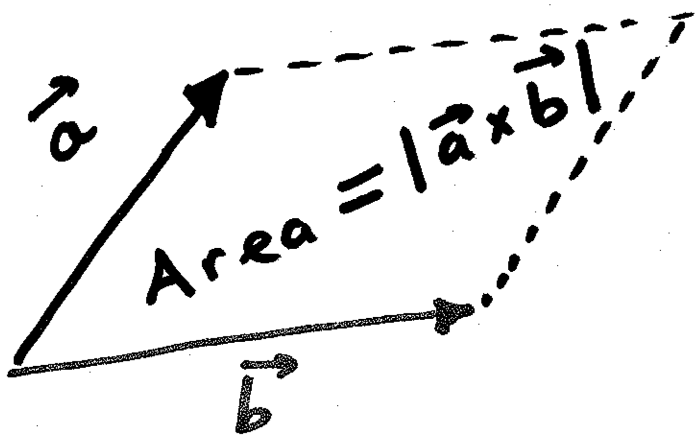
(ii) Parallel

$\vec{a}$ and $\vec{b}$ are parallel 3D vectors if: $\vec{a} \times \vec{b} = \vec{0}.$

(b) Area

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\theta)$$

- If $\vec{a}$ and $\vec{b}$ are 3D vectors, the area of the parallelogram that has $\vec{a}$ and $\vec{b}$ as its sides is: $|\vec{a} \times \vec{b}|$.

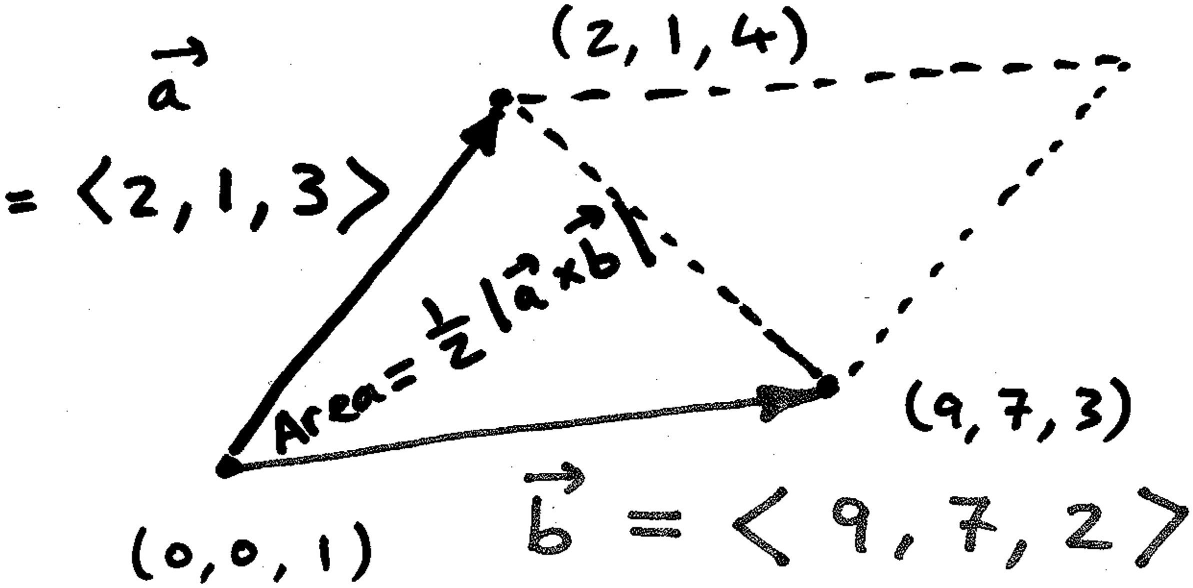


Example

Find the area of the triangle with vertices at:

$(0,0,1) \quad (2,1,4) \quad (9,7,3)$.

Solution



$$\vec{a} \times \vec{b} = \begin{array}{ccccc} i & j & k & i & j \\ 2 & 1 & 3 & 2 & 1 \\ 9 & 7 & 2 & 9 & 7 \end{array}$$

$$= i(1)(2) + j(3)(9) + k(2)(7)$$

$$- j(2)(2) - i(3)(7) - k(1)(9)$$

$$= -19\vec{i} + 23\vec{j} + 5\vec{k}$$

$$= \langle -19, 23, 5 \rangle.$$

$$\text{Area} = \frac{1}{2}\sqrt{19^2 + 23^2 + 5^2} \approx 15.12.$$

Example

Find the equation of the plane that contains the points:

$(0,0,1)$ $(2,1,4)$ $(9,7,3)$.

Solution

$$a\cdot(x-x_0)+b\cdot(y-y_0)+c\cdot(z-z_0)=0$$

where: $\langle a,b,c\rangle$ is the normal vector.

(x_0, y_0, z_0) is a point on the plane.

Need: Normal vector.

From previous example,

$\vec{a}=\langle 2,1,3\rangle$ $\vec{b}=\langle 9,7,2\rangle$

both lie in the plane.

Normal vector $= \vec{a} \times \vec{b}$

$= \langle -19, 23, 5 \rangle.$

Equation of plane:

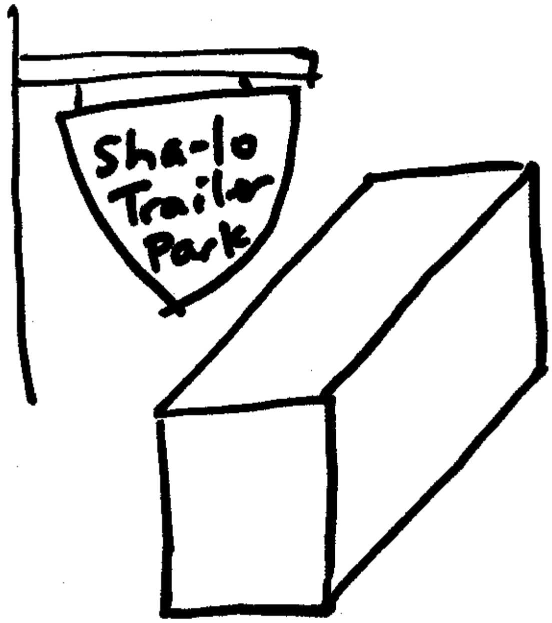
$\langle a, b, c \rangle = \langle -19, 23, 5 \rangle$

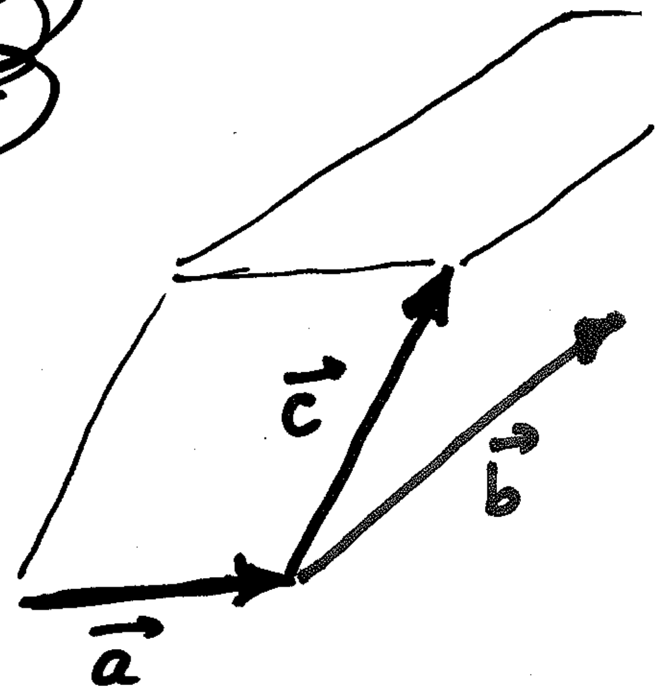
$(x_0, y_0, z_0) = (2, 1, 4)$

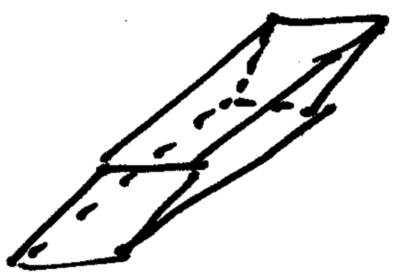
$$-19(x-2) + 23(y-1) + 5(z-4) = 0.$$

2. Parallelpipeds

- A parallelpiped is a 3D shape that looks like a trailer after a tornado has hit.







- The volume enclosed by the parallelpiped with edges described by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

$$V = \left| \vec{a} \bullet (\vec{b} \times \vec{c}) \right|.$$

Example

Find the volume of the parallelpiped with sides PQ, PR and PS where:

$P = (3,0,1)$ $\qquad$ $R = (5,1,-1)$

$Q = (-1,2,5)$ $\qquad$ $S = (0,4,2)$

Solution

$$\vec{a} = \overrightarrow{PQ} = \langle -4, 2, 4 \rangle$$

$$\vec{b} = \overrightarrow{PR} = \langle 2, 1, -2 \rangle$$

$$\vec{c} = \overrightarrow{PS} = \langle -3, 4, 1 \rangle$$

$$V = |\vec{a} \bullet (\vec{b} \times \vec{c})|.$$

$$\vec{b} \times \vec{c} = \begin{matrix} i & j & k & i & j \\ 2 & 1 & -2 & 2 & 1 \\ -3 & 4 & 1 & -3 & 4 \end{matrix}$$

$\vec{b} \times \vec{c} = \langle 9, 4, 11 \rangle$

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = \langle -4, 2, 4 \rangle \bullet \langle 9, 4, 11 \rangle$$
$$= 16.$$

Volume $= 16$ units.

3. Lines and Planes in 3D

Line in 3D:

$$\langle x, y, z \rangle = \underbrace{\langle x_0, y_0, z_0 \rangle}_{\text{point on line}} + t \cdot \underbrace{\langle a, b, c \rangle}_{\text{direction vector of line.}}$$

Plane in 3D:

$$\underbrace{\langle a, b, c \rangle}_{\text{normal vector}} \bullet \left(\langle x, y, z \rangle - \underbrace{\langle x_0, y_0, z_0 \rangle}_{\text{point known to be on plane.}} \right) = 0$$

$$\langle a, b, c \rangle \bullet \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

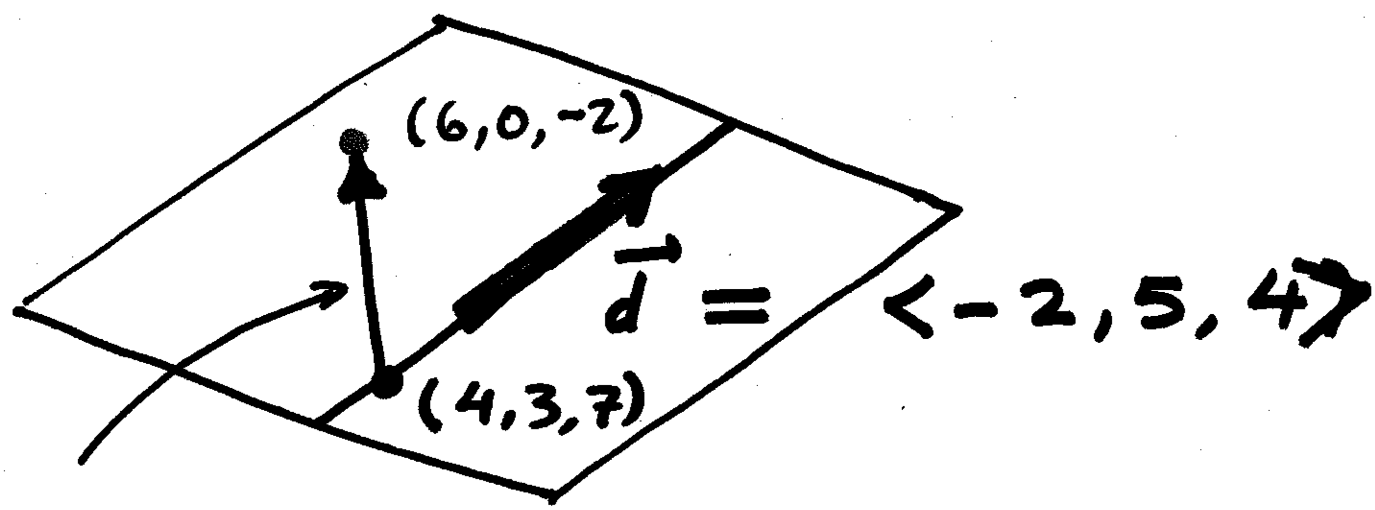
Example

Find an equation for the plane that contains:

Point: $(6, 0, -2)$

Line: $\langle x, y, z \rangle = \langle 4, 3, 7 \rangle + t \cdot \langle -2, 5, 4 \rangle.$

Solution



$\vec{p} = \langle 2, -3, -9 \rangle$

Normal vector $= \vec{d} \times \vec{p}$

$$= \begin{array}{ccccc} i & j & k & i & j \\ -2 & 5 & 4 & -2 & 5 \\ 2 & -3 & -9 & 2 & -3 \end{array}$$

$$= \langle -45 + 12,\ 8 - 18,\ 6 - 10 \rangle$$

$$= \langle -33,\ -10,\ -4 \rangle$$

Equation of plane:

$$-33(x-6) - 10(y-0) - 4(z+2) = 0.$$

4. Symmetric Equations.

- Equations that specify a line in 3D.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a point on the line

$\langle a, b, c \rangle$ is the direction vector of the line.

Example

Find the symmetric equation for:

$$\langle x, y, z \rangle = \underbrace{\langle 3, 4, 7 \rangle}_{x_0, y_0, z_0} + t \cdot \underbrace{\langle 12, 9, -5 \rangle}_{a, b, c}.$$

Solution

$$x = 3 + 12t \Rightarrow t = \frac{x-3}{12}$$

$$y = 4 + 9t \Rightarrow t = \frac{y-4}{9}$$

$$z = 7 - 5t \Rightarrow t = \frac{z-7}{-5}$$

$$\frac{x-3}{12} = \frac{y-4}{9} = \frac{z-7}{-5}.$$